The 31st Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13th April 2024
Category II

Problem 1 *Suppose that* $f\colon [-1,1] \to \mathbb{R}$ *is continuous and that*

$$\left(\int_{-1}^{1} \mathrm{e}^{x} f(x)\,\mathrm{d}x\right)^2 \geq \left(\int_{-1}^{1} f(x)\,\mathrm{d}x\right)\left(\int_{-1}^{1} \mathrm{e}^{2x} f(x)\,\mathrm{d}x\right).$$

Prove that there exists a point $c \in (-1,1)$ *such that* $f(c) = 0$.

[Robert Skiba / Nicolaus Copernicus University in Toruń]

Solution Assume on the contrary that $f(x) \neq 0$ for all $x \in (-1,1)$. Then $f(x)$ must be everywhere positive or negative. By replacing $f(x)$ with $-f(x)$ if necessary, we can assume that $f(x) > 0$ on $(-1,1)$. Then we can write

$$f(x) = \left(\sqrt{f(x)}\right)^2.$$

Hence, we get

$$\left(\int_{-1}^{1} \mathrm{e}^{x} \left(\sqrt{f(x)}\right)^2 \mathrm{d}x\right)^2 \geq \left(\int_{-1}^{1} f(x)\mathrm{d}x\right)\left(\int_{-1}^{1} \mathrm{e}^{2x} f(x)\mathrm{d}x\right). \tag{1}$$

On the other hand, the Cauchy-Schwarz inequality implies that

$$\left(\int_{-1}^{1} \mathrm{e}^{x} \left(\sqrt{f(x)}\right)^2 \mathrm{d}x\right)^2 = \left(\int_{-1}^{1} \left(\mathrm{e}^{x}\sqrt{f(x)}\right)\sqrt{f(x)}\mathrm{d}x\right)^2 \leq \left(\int_{-1}^{1} \mathrm{e}^{2x} f(x)\mathrm{d}x\right)\left(\int_{-1}^{1} f(x)\mathrm{d}x\right). \tag{2}$$

Taking into account (1) and (2), we get

$$\left(\int_{-1}^{1} \mathrm{e}^{x}\sqrt{f(x)}\sqrt{f(x)}\mathrm{d}x\right)^2 = \left(\int_{-1}^{1} f(x)\mathrm{d}x\right)\left(\int_{-1}^{1} \mathrm{e}^{2x} f(x)\mathrm{d}x\right).$$

On the other hand, it is well known that the equality holds in the Cauchy-Schwarz inequality if and only if $\mathrm{e}^{x}\sqrt{f(x)}$ is a constant multiple of $\sqrt{f(x)}$, but this is not possible. Therefore, we can conclude, by a contradiction argument, that there exists a point $c \in (-1,1)$ such that $f(c) = 0$. □

Problem 2 *A real 2024×2024 matrix A is called nice if $(Av, v) = 1$ for every vector $v \in \mathbb{R}^{2024}$ with unit norm.*

a) Prove that the only nice matrix such that all of its eigenvalues are real is the identity matrix.

b) Find an example of a nice non-identity matrix. [Stoyan Apostolov / Sofia University]

Solution Using the properties of transposed matrices, we obtain:

$$2(Av, v) = (Av, v) + (v, Av) = (Av, v) + (A^T v, v) = ((A + A^T)v, v) = 2 \tag{1}$$

for every unit vector v. Since $A + A^T$ is symmetric, all eigenvalues of $A + A^T$ are real. From (1), it follows that all eigenvalues of $A + A^T$ are equal to 2. But every symmetric matrix is diagonalizable, therefore $A + A^T$ is similar to a scalar matrix with 2 along the diagonal, the matrix $2I$ (where I denotes the identity matrix of order n). It is directly seen that any matrix similar to a scalar matrix is also scalar. Thus, $A + A^T = 2I$. Consequently A is normal. Since its characteristic roots are real, it is Hermitian and hence symmetric. Thus, from $A + A^T = 2I$, we obtain $A = I$.

b) Let B be a nonzero antisymmetric matrix. It is directly verified that $(Bv, v) = 0$ for every vector v. Then $A := B + I$ is non-identity and satisfies the condition of the problem. □

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Problem 3 *Let $a_1 > 0$ and for $n \geq 1$ define*

$$a_{n+1} = a_n + \frac{1}{a_1 + a_2 + \ldots + a_n}.$$

Prove that $\lim_{n\to\infty} \frac{a_n^2}{\ln n} = 2.$ [Teodor Chelmuș / Alexandru Ioan Cuza University of Iași]

Solution Since $a_1 > 0$, it follows that the given sequence is strictly nondecreasing. Let $\ell \in (0, \infty]$ the limit of the sequence $(a_n)_{n\in\mathbb{N}^*}$. If ℓ would be finite, then

$$\frac{1}{\ell} = \lim_{n\to\infty} \frac{1}{a_n} = \lim_{n\to\infty} \frac{n}{a_1 + a_2 + \ldots + a_n} = \lim_{n\to\infty} n(a_{n+1} - a_n).$$

Using the telescoping technique, and the limit above, one has

$$\ell - a_1 = \lim_{n\to\infty} a_n - a_1 = \sum_{n=1}^{\infty} (a_{n+1} - a_n) \sim \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Contradiction. So $a_n \to \infty$. Further we will prove that that a_n goes to infinity in same manner as the sequence $(\sqrt{2\ln n})_{n\in\mathbb{N}^*}$ does. The presence of the $\ln n$ suggests to us to think at harmonic series and the fact that

$$\lim_{n\to\infty} \frac{1}{\ln n}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) = 1.$$

It is enough to show that

$$\lim_{n\to\infty} \frac{a_n^2}{1 + \frac{1}{2} + \ldots + \frac{1}{n}} = \lim_{n\to\infty} \frac{a_n^2 - a_1^2}{1 + \frac{1}{2} + \ldots + \frac{1}{n}} = 2$$

Let $S_n = a_1 + a_2 + \ldots + a_n$. We will use, again, the telescoping technique to write that

$$\lim_{n\to\infty} a_n^2 - a_1^2 = \sum_{n=1}^{\infty} (a_{n+1}^2 - a_n^2) = \sum_{n=1}^{\infty} \frac{a_{n+1} + a_n}{S_n} \tag{1}$$

Taking into account that $a_{n+1} - a_n = \frac{1}{S_n}$, we have

$$a_{n+1}^2 - a_n^2 = \frac{a_{n+1} + a_n}{S_n} = \frac{a_n}{S_n}\left(\frac{a_{n+1}}{a_n} + 1\right) \tag{2}$$

Observe now that

$$\frac{S_n}{a_n} = \frac{S_{n-1} + a_n}{a_n} = \frac{S_{n-1}}{a_n} + 1 \implies \frac{S_n}{a_n} - \frac{S_{n-1}}{a_{n-1}} = 1 + \frac{S_{n-1}}{a_n} - \frac{S_{n-1}}{a_{n-1}} = 1 + \frac{1}{a_n a_{n-1}}.$$

Passing to limit, the sequence $(S_n/a_n - S_{n-1}/a_{n-1})$ is convergent to 1, and using, again, that if a sequence admits a limits (finite or not), then the mean values sequence (Cesaro mean) admits the same limit, we deduce that

$$1 = \lim_{n\to\infty}\left(\frac{S_n}{a_n} - \frac{S_{n-1}}{a_{n-1}}\right) = \lim_{p\to\infty} \frac{1}{p}\sum_{n=1}^{p}\left(\frac{S_n}{a_n} - \frac{S_{n-1}}{a_{n-1}}\right) = \lim_{p\to\infty} \frac{S_p}{p a_p}.$$

Going back in (2), and using that $\frac{a_{n+1}}{a_n} \to 1$, is follows that

$$\lim_{n\to\infty} n(a_{n+1}^2 - a_n^2) = \lim_{n\to\infty} \frac{n a_n}{S_n}\left(\frac{a_{n+1}}{a_n} + 1\right) = 2.$$

The proof is complete. □

Problem 4 *Let $(b_n)_{n\geq 0}$ be a sequence of positive integers satisfying $b_n = d\left(\sum_{k=0}^{n-1} b_k\right)$ for all $n \geq 1$. (By $d(m)$ we denote the number of positive divisors of m.)*

a) Prove that $(b_n)_{n\geq 0}$ is unbounded.

b) Prove that there are infinitely many n such that $b_n > b_{n+1}$. [Adrian Beker / University of Zagreb]

Solution Define $s_n = \sum_{k=0}^{n-1} a_k$ for $n \geq 0$. Thus, $(s_n)_{n\geq 0}$ is a strictly increasing sequence such that $s_0 = 0$. Moreover, $a_n = d(s_n)$ for all $n \geq 1$.

(i) Suppose for contradiction that there exists $C \in \mathbb{N}$ such that $a_n \leq C$ for all $n \geq 0$. Enumerate the primes as a strictly increasing sequence $(p_k)_{k\geq 1}$. By the Chinese Remainder Theorem, there exists a positive integer x such that $x \equiv -j \pmod{p_j^C}$ for all $1 \leq j \leq C$. In particular, we have $d(x+j) \geq C+1$ for all $1 \leq j \leq C$. Now choose the least $n \geq 0$ such that $s_n > x$. Then we must have $n \geq 1$, so by minimality of n, we have $s_{n-1} \leq x$. Thus,

$$x < s_n = s_{n-1} + a_{n-1} \leq x + C,$$

so it follows that $a_n = d(s_n) > C$, which is a contradiction.

(ii) We begin by establishing the following auxiliary result:

Lemma *Given a positive integer a, let $f(a)$ be the length of the longest arithmetic progression of positive integers with common difference a all of whose terms have exactly a divisors. Then we have $f(a) \ll_\varepsilon a^{1+\varepsilon}$ for any $\varepsilon > 0$.*

Proof We may assume that ε is small and fixed and a is large. Enumerate the primes and the primes not dividing a as strictly increasing sequences $(p_k)_{k\geq 1}$ and $(q_k)_{k\geq 1}$ respectively. Then we have $q_k \leq p_{k+\omega(a)}$ for all $k \geq 1$. Fix $k \geq 1$, write $\ell = v_{p_k}(a)$ and consider the number $b = \prod_{j=1}^{\ell+1} q_j^{p_k}$. We claim that $f(a) < b$. Indeed, consider any arithmetic progression $s, s+a, \ldots, s+(b-1)a$ of length b with common difference a. Since a and b are coprime, it follows that $\{0, a, \ldots, (b-1)a\}$ is a complete residue system modulo b, and hence so is $\{s, s+a, \ldots, s+(b-1)a\}$. In particular, by the Chinese Remainder Theorem, there exists $i \in \{0, 1, \ldots, b-1\}$ such that $s + ia \equiv q_j^{p_k - 1} \pmod{q_j^{p_k}}$ for all $1 \leq j \leq \ell+1$. But this means that $v_{q_j}(s+ia) = p_k - 1$ for all $1 \leq j \leq \ell+1$ and hence that $p_k^{\ell+1} \mid d(s+ia)$. In particular, we cannot have $d(s+ia) = a$, so the claim follows. It remains to find a good upper bound on b for various values of k.

Suppose that $f(a) \geq a^{1+\varepsilon}$. Since $b \leq q_{\ell+1}^{(\ell+1)p_k} \leq p_{\ell+\omega(a)+1}^{(\ell+1)p_k}$, it follows by taking logarithms that $(\ell+1)\log p_{\omega(a)+\ell+1} \geq \frac{1+\varepsilon}{p_k}\log a$. By a weak version of the prime number theorem, we have $\pi(x) = \Omega\left(\frac{x}{\log x}\right)$ for $x \geq 2$, so it follows that $p_m = \mathcal{O}(m \log m)$ for $m \geq 2$. Thus, $\log p_m \leq \log m + \log\log m + \mathcal{O}(1)$ for $m \geq 2$, so $\log p_m \leq \left(1 + \frac{\varepsilon}{6}\right)\log m$ if m is large enough. On the other hand, it is clear that $\omega(a), \ell \leq \log_2 a$, so $m = \omega(a) + \ell + 1$ satisfies $m \leq 2\log_2 a + 1 \leq 6 \log a$ if $a \geq 2$. Hence, if a is large enough, it follows that $\log p_m \leq \left(1+\frac{\varepsilon}{3}\right)\log\log a$, whence $\ell + 1 \geq \frac{1+\frac{\varepsilon}{3}}{p_k}\frac{\log a}{\log\log a}$ if $\varepsilon \in (0,3)$. Therefore, letting $x = \frac{1+\frac{\varepsilon}{3}}{1+\frac{9}{\varepsilon}}\frac{\log a}{\log\log a}$, if $p_k \leq x$, it follows that $\ell \geq \frac{9}{\varepsilon}$ and hence that $\ell \geq \frac{\ell+1}{1+\frac{\varepsilon}{9}} \geq \frac{1+\frac{\varepsilon}{9}}{p_k}\frac{\log a}{\log\log a}$. Therefore, we have

$$\log a \geq \sum_{p_k \leq x} v_{p_k}(a)\log p_k \geq \left(1+\frac{\varepsilon}{9}\right)\frac{\log a}{\log\log a}\sum_{p_k\leq x}\frac{\log p_k}{p_k}.$$

But by Mertens' first theorem, we have $\sum_{p_k \leq x}\frac{\log p_k}{p_k} = \log x + \mathcal{O}(1)$, so it follows that $x \ll (\log a)^{\frac{1}{1+\frac{\varepsilon}{9}}}$, which is a contradiction if a is large. Thus, the lemma is proved. □

It is now not hard to prove the desired statement. Indeed, it is a standard fact that, for any $\delta > 0$, we have $d(m) \ll_\delta m^\delta$. Hence, we have $d(m) \leq m^{\frac{1}{5}}$ for all sufficiently large m. Now consider the function

$$g : (0,\infty) \to \mathbb{R}, \quad t \mapsto t^{\frac{4}{5}}.$$

Then g is differentiable with $g'(t) = \frac{4}{5}t^{-\frac{1}{5}}$, which is a decreasing function. By the Mean Value Theorem, for all sufficiently large n we have

$$g(s_{n+1}) - g(s_n) \leq (s_{n+1} - s_n)g'(s_n) = d(s_n)g'(s_n) \leq s_n^{\frac{1}{5}} \cdot \frac{4}{5}s_n^{-\frac{1}{5}} = \frac{4}{5}.$$

It follows that $g(s_n) \ll n$, whence $s_n \ll n^{\frac{5}{4}}$ and hence there is a constant B such that $a_n \leq Bn^{\frac{1}{4}}$ for all $n \geq 1$. Now suppose for contradiction that there exists $N \geq 0$ such that $a_n \leq a_{n+1}$ for all $n > N$. By the Lemma for $\varepsilon = 1$, it follows that for each $a \in \mathbb{N}$ there are at most Ca^2 integers $n > N$ such that $a_n = a$, where C is some absolute constant. It now follows that $C \sum_{a \leq BM^{\frac{1}{4}}} a^2 \geq M - N$ for all $M > N$, which is a contradiction for large M since $\sum_{a \leq x} a^2 = \mathcal{O}(x^3)$. $\square$