The 30th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 2nd April 2022
Category II

Problem 1 *Determine whether there exists a differentiable function* $f\colon [0,1] \to \mathbb{R}$ *such that*

$$f(0) = f(1) = 1\,, \quad |f'(x)| \leq 2 \text{ for all } x \in [0,1] \quad \text{and} \quad \left| \int_0^1 f(x)\,\mathrm{d}x \right| \leq \frac{1}{2}\,.$$

[Robert Skiba / Nicolaus Copernicus University in Toruń]

Solution Let us suppose that there exists the required function $f\colon [0,1] \to \mathbb{R}$. Then the mean value theorem implies that for any $x \in (0,1)$ there exist $c_x, d_x \in (0,1)$ such that

$$\frac{f(x) - f(0)}{x} = \frac{f(x) - 1}{x} = f'(c_x) \geq -2 \Longrightarrow f(x) \geq -2x + 1$$

and

$$\frac{f(x) - f(1)}{x - 1} = \frac{f(x) - 1}{x - 1} = f'(d_x) \leq 2 \Longrightarrow f(x) \geq 2x - 1.$$

Hence we infer that

$$f(x) \geq |2x - 1|$$

for all $x \in [0,1]$. Thus

$$\int_0^1 f(x)\,\mathrm{d}x \geq \int_0^1 |2x - 1|\,\mathrm{d}x = \frac{1}{2}.$$

On the other hand, one has

$$\int_0^1 f(x)\,\mathrm{d}x \leq \left| \int_0^1 f(x)\,\mathrm{d}x \right| \leq \frac{1}{2}.$$

Thus

$$\int_0^1 f(x)\,\mathrm{d}x = \frac{1}{2},$$

which implies that

$$\int_0^1 (f(x) - |2x - 1|)\,\mathrm{d}x = 0.$$

But $f(x) - |2x - 1|$ is nonnegative and continuous and therefore $f(x) - |2x - 1| = 0$ for all $x \in [0,1]$. Hence $f(x) = |2x - 1|$ for all $x \in [0,1]$. This implies that $f(x)$ is not differentiable at the point $x = \frac{1}{2}$, which implies the contradiction because we assumed that $f(x)$ is differentiable at any point $x \in [0,1]$. $\square$

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Problem 2 *For any given pair of positive integers $m > n$ find all $a \in \mathbb{R}$ for which the polynomial $x^m - ax^n + 1$ can be expressed as a quotient of two nonzero polynomials with real nonnegative coefficients.*

[Artūras Dubickas / Vilnius University]

Solution The answer is $a < m \cdot n^{-n/m} \cdot (m-n)^{n/m-1}$.

For $a \le 0$ the given polynomial
$$f(x) = x^m - ax^n + 1 \tag{1}$$
has the required form $P(x)/Q(x)$, since one can select $P(x) = f(x)$ and $Q(x) = 1$. So, from now on, let us assume that $a > 0$.

Note that $f'(x) = mx^{m-1} - anx^{n-1}$ vanishes at $x_0 = (an/m)^{1/(m-n)}$. Thus, the polynomial f has a real positive root, say β, if
$$f(x_0) = 1 - x_0^n(a - x_0^{m-n}) = 1 - (m-n)n^{n/(m-n)}(a/m)^{m/(m-n)} \le 0,$$
i.e., $a \ge m \cdot n^{-n/m} \cdot (m-n)^{n/m-1}$. For any such a assume that one has $f(x) = P(x)/Q(x)$ with appropriate polynomials P and Q. Then, $P(\beta) = f(\beta)Q(\beta) = 0$, which is impossible, since $\beta > 0$ and $P \in \mathbb{R}[x]$ has real nonnegative coefficients (and is nonzero).

Finally, we will show that $f(x)$ can be expressed in the required form $P(x)/Q(x)$ if a satisfies the opposite inequality
$$0 < a < m \cdot n^{-n/m} \cdot (m-n)^{n/m-1}. \tag{2}$$
To prove this we will use the formula $u^K - v^K = (u-v)(u^{K-1} + \cdots + v^{K-1})$ with $u = x^m + 1$, $v = ax^n$ and K divisible by mn. Fix any a satisfying (2). Set
$$P(x) = (x^m + 1)^{mnN} - (ax^n)^{mnN}, \quad Q(x) = \sum_{j=0}^{mnN-1} (x^m + 1)^{mnN-1-j}(ax^n)^j$$
with $N \in \mathbb{N}$ to be chosen later. Then, $f(x) = P(x)/Q(x)$ by (1). Evidently, the polynomial Q has real nonnegative coefficients. So does also P if the binomial coefficient $\binom{mnN}{n^2N}$ for x^{mn^2N} (in $(x^m+1)^{mnN}$) is greater than or equal to a^{mnN}. Thus, it remains to check that
$$a \le \binom{mnN}{n^2N}^{1/(mnN)} = \left(\frac{(mnN)!}{((m-n)nN)!(n^2N)!}\right)^{1/(mnN)} \tag{3}$$
for some $N \in \mathbb{N}$. By Stirling's formula, $M! \sim \sqrt{2\pi M}(M/e)^M$ as $M \to \infty$. Hence, as $N \to \infty$, the right hand side of (3) tends to the constant
$$\frac{mn}{\big((m-n)n\big)^{(m-n)/m} n^{2n/m}} = m \cdot n^{-n/m} \cdot (m-n)^{n/m-1}.$$
Therefore, by (2), the inequality (3) holds for each sufficiently large $N \in \mathbb{N}$. $\square$

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Problem 3 *Let $x_1,\dots,x_n$ be given real numbers with $0<m\le x_i\le M$ for each $i\in\{1,\dots,n\}$. Let X be the discrete random variable uniformly distributed on $\{x_1,\dots,x_n\}$. The mean μ and the variance σ^2 of X are defined as*

$$\mu(X)=\frac{x_1+\cdots+x_n}{n}\quad\text{and}\quad \sigma^2(X)=\frac{(x_1-\mu(X))^2+\cdots+(x_n-\mu(X))^2}{n}.$$

By X^2 denote the discrete random variable uniformly distributed on $\{x_1^2,\dots,x_n^2\}$. Prove that

$$\sigma^2(X)\ge\left(\frac{m}{2M^2}\right)^2\sigma^2(X^2).$$

[Slobodan Filipovski / University of Primorska, Koper]

Solution First we prove the following lemma:

Lemma *If x and y are strictly positive real numbers, then*

$$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\ge 2+\frac{(x-y)^2}{2(x^2+y^2)}.$$

Proof We prove the following equivalent inequality

$$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}\ge 2+\frac{\left(\frac{x}{y}\right)^2-2\left(\frac{x}{y}\right)+1}{2\left(\left(\frac{x}{y}\right)^2+1\right)}.$$

Let $t^2=\frac{x}{y}$, $t>0$. The required inequality is equivalent to the inequalities

$$t+\frac{1}{t}\ge 2+\frac{t^4-2t^2+1}{2(t^4+1)}\Leftrightarrow 2t^6-5t^5+2t^4+2t^3+2t^2-5t+2\ge 0.$$

Now we easily show $2t^6-5t^5+2t^4+2t^3+2t^2-5t+2=(t-1)^4(2t^2+3t+2)\ge 0$. $\square$

Let $a_i=\frac{x_i^2}{x_1^2+\ldots+x_n^2}$ and $b_i=\frac{1}{n}$ for $i=1,\dots,n$. Applying the above lemma for $x=a_i$ and $y=b_i$ we obtain

$$\frac{x_i^2}{x_1^2+\ldots+x_n^2}+\frac{1}{n}\ge\left(2+\frac{(x_i^2n-(x_1^2+\ldots+x_n^2))^2}{2(x_i^4n^2+(x_1^2+\ldots+x_n^2)^2)}\right)\frac{x_i}{\sqrt{(n(x_1^2+\ldots+x_n^2)}}. \tag{1}$$

Now if we sum up the obtained n inequalities in (1) we get

$$2\ge\frac{2}{\sqrt{n(x_1^2+\ldots+x_n^2)}}\sum_{i=1}^n x_i+\frac{m}{\sqrt{n(x_1^2+\ldots+x_n^2)}}\cdot\frac{1}{2(M^4+\mu^2(X^2))}\cdot\sum_{i=1}^n(x_i^2-\frac{x_1^2+\ldots+x_n^2}{n})^2\Leftrightarrow$$
$$\sqrt{\frac{x_1^2+\ldots+x_n^2}{n}}\ge\frac{\sum_{i=1}^n x_i}{n}+\frac{m\cdot\sigma^2(X^2)}{4(M^4+\mu^2(X^2))}=\mu(X)+\frac{m\cdot\sigma^2(X^2)}{4(M^4+\mu^2(X^2))}\Leftrightarrow$$
$$\sqrt{\mu(X^2)}\ge\mu(X)+\frac{m\cdot\sigma^2(X^2)}{4(M^4+M^4)}=\mu(X)+\frac{m\cdot\sigma^2(X^2)}{8M^4}.$$

In the end we get

$$\sigma^2(X)=\left(\sqrt{\mu(X^2)}-\mu(X)\right)\left(\sqrt{\mu(X^2)}+\mu(X)\right)\ge\frac{m\sigma^2(X^2)}{8M^4}\cdot 2m=\left(\frac{m}{2M^2}\right)^2\cdot\sigma^2(X^2).$$

$\square$

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Problem 4 *A function $f\colon \mathbb{Z}^+ \to \mathbb{R}$ is called* multiplicative *if for every $a, b \in \mathbb{Z}^+$ with $\gcd(a,b) = 1$ we have $f(ab) = f(a)f(b)$. Let g be the multiplicative function given by*

$$g(p^\alpha) = \alpha p^{\alpha-1}\,,$$

where $\alpha \in \mathbb{Z}^+$ and $p > 0$ is a prime. Prove that there exist infinitely many positive integers n such that

$$g(n) + 1 = g(n+1)\,.$$

[Leonhard Summerer / University of Vienna]

Solution First we observe that $g(n) = 1$ for all squarefree integers n. Then we start by finding integers a and b for which $g(a) + 1 = g(b)$. For example $a = 13^2$ and $b = 3^3$ so that $g(a) = 26$ and $g(b) = 27$. By the observation at the beginning combined with the multiplicativity of g we have $g(ax) = 26$ and $g(by) = 27$ provided x, y are squarefree positive integers with $(x, 13) = (y, 3) = 1$. It thus suffices to show the existence of at least one (resp. infinitely many) solution(s) of the linear diophantine equation $ax - by = -1$ with the mentioned restrictions on x, y.

It is well known that all solutions of the above equation are given by

$$x = x_0 + 27t \text{ and } y = y_0 + 169t,$$

where (x_0, y_0) is a particular solution and $t = 0, 1, 2, \ldots$. Using the Euclidean Algorithm, one easily finds that the least positive solution is given by $x_0 = 23$ and $y_0 = 144$. Unfortunately 144 is neither squarefree nor coprime to 3, but for $t = 2$ we find $x = 77 = 7 \cdot 11$ and $y = 482 = 2 \cdot 241$ which fulfill all requirements and lead to the solution

$$g(13013) = 26 \text{ and } g(13014) = 27.$$

In order to find infinitely many solutions we consider the sequences

$$x_s = 77 + 27s \text{ and } y_s = 482 + 169s,$$

where $s = 39t$, $t = 0, 1, 2, \ldots$ which guarantees $(x, 13) = (y, 3) = 1$. It suffices to show that there exist infinitely many s such that x_s and y_s are simultaneously squarefree. This follows from a Theorem of Prachar, saying that the density of squarefree integers in the arithmetic progression $nk + l$ where $(k, l) = 1$ is

$$\frac{6}{\pi^2} \prod_{p \mid k} \left(1 - \frac{1}{p^2}\right)^{-1},$$

which is always greater than $6/\pi^2$ and hence greater than $1/2$. $\square$