The 30$^{\text{th}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 2$^{\text{nd}}$ April 2022
Category I

Problem 1 *Assume that a real polynomial $P(x)$ has no real roots. Prove that the polynomial*

$$Q(x) = P(x) + \frac{P''(x)}{2!} + \frac{P^{(4)}(x)}{4!} + \dots$$

also has no real roots. [Diana Barseghyan / University of Ostrava]

Solution Since P at any point coincides with its Taylor series one has

$$P(x+1) = P(x) + P'(x) + \frac{P''(x)}{2!} + \dots$$
$$P(x-1) = P(x) - P'(x) + \frac{P''(x)}{2!} - \dots$$

Let us notice that

$$Q(x) = \frac{P(x+1) + P(x-1)}{2}.$$

Hence $Q(x)$ preserves sign together with $P(x)$ which finishes the proof. □

Problem 2 *Let $n \geq 1$. Assume that A is a real $n \times n$ matrix which satisfies the equality*

$$A^7 + A^5 + A^3 + A - I = 0\,.$$

Show that $\det(A) > 0$. [Daniel Strzelecki / Nicolaus Copernicus University in Toruń]

Solution Put $w(x) = x^7 + x^5 + x^3 + x - 1$. Since $w(A) = 0$, any eigenvalue of A is a root of $w(x)$. We are going to show that only one eigenvalue is real and strictly positive and the remaining eigenvalues belong to the set $\mathbb{C} \setminus \mathbb{R}$. Since

$$w'(x) = 7x^6 + 5x^4 + 3x^2 + 1 > 0$$

for all $x \in \mathbb{R}$, it follows that $w(x)$ has only one real root. The remaining roots of $w(x)$ are pairs of complex conjugate numbers. Furthermore, we have $w(0) = -1$ and $w(1) = 3$, so $w(\lambda_0) = 0$ for some $\lambda_0 \in (0, 1)$ i.e. $w(x)$ possesses exactly one positive root.

It is well known that the determinant is a product of all (not necessarily different eigenvalues). So, finally, we obtain

$$\det(A) = \lambda_0^{\alpha_0} \lambda_1^{\alpha_1} \overline{\lambda_1^{\alpha_1}} \cdot \ldots \cdot \lambda_m^{\alpha_m} \overline{\lambda_m^{\alpha_m}} = \lambda_0^{\alpha_0} \cdot |\lambda_1|^{\alpha_1} \cdot \ldots |\lambda_m|^{\alpha_m} > 0,$$

where $\lambda_1, \ldots, \lambda_m \in \sigma(A) \cap \mathbb{C}$ with $\alpha_i \in \mathbb{N} \cup \{0\}$. This completes the solution. $\square$

The 30th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 2nd April 2022
Category I

Problem 3 *Let $f\colon [0,1] \to \mathbb{R}$ be a given continuous function. Find the limit*

$$\lim_{n\to\infty}(n+1)\sum_{k=0}^{n}\int_0^1 x^k(1-x)^{n-k}f(x)\,\mathrm{d}x\,.$$

[Marcin Zygmunt / University of Silesia in Katowice]

Solution The limit is equal to $f(0)+f(1)$.

First we need the following lemma:

Lemma. *Let $f : [0,1] \to \mathbb{R}$ be an integrable function, continuous at $x = 1$. Then*

$$\lim_{n\to\infty}(n+1)\int_0^1 x^n f(x)\,\mathrm{d}x = f(1)\,.$$

Proof We will show that the difference of the integral and $f(1)$ tends to 0 as n goes to infinity. Let $\varepsilon > 0$. Then (by the continuity at $x = 1$) there exists $\delta > 0$ such that $|f(x) - f(1)| < \varepsilon/2$ for $x \in (1-\delta, 1]$. As $0 < 1-\delta < 1$ then there exists n_0 such that $(n+1)(1-\delta)^n < \varepsilon/2(M + |f(1)|)$ for all $n \geq n_0$, where $M = \int_0^1 |f(x)|\,\mathrm{d}x$.

We have

$$(n+1)\int_0^1 x^n f(x)\,\mathrm{d}x - f(1) = \int_0^1 x^n\bigl(f(x)-f(1)\bigr)\,\mathrm{d}x\,,$$

hence

$$\begin{aligned}
\left|(n+1)\int_0^1 x^n\bigl(f(x)-f(1)\bigr)\,\mathrm{d}x\right| &\leq (n+1)\int_0^1 x^n\bigl|f(x)-f(1)\bigr|\,\mathrm{d}x\\
&= (n+1)\int_0^{1-\delta}\cdots + (n+1)\int_{1-\delta}^1\cdots\\
&\leq (n+1)(1-\delta)^n\int_0^{1-\delta}\bigl(|f(x)|+|f(1)|\bigr)\,\mathrm{d}x + \frac{\varepsilon}{2}(n+1)\int_{1-\delta}^1 x^n\,\mathrm{d}x\\
&\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon\,,
\end{aligned}$$

for all $n \geq n_0$. This, by definition, means that

$$\lim_{n\to\infty}(n+1)\int_0^1 x^n f(x)\,\mathrm{d}x = f(1)\,.$$

□

Now we can proceed with the solution to this specific limit. Let us assume for the moment that all following formulas and transformations make sense. So we have

$$\begin{aligned}
(n+1)\sum_{k=0}^{n}\int_0^1 x^k(1-x)^{n-k}f(x)\,\mathrm{d}x &= (n+1)\int_0^1\frac{1}{2x-1}\bigl(x-(1-x)\bigr)\Bigl(\sum_{k=0}^{n}x^k(1-x)^{n-k}\Bigr)f(x)\,\mathrm{d}x\\
&= (n+1)\int_0^1\frac{x^{n+1}-(1-x)^{n+1}}{2x-1}f(x)\,\mathrm{d}x\\
&= (n+1)\int_0^1\frac{x^{n+1}}{2x-1}f(x)\,\mathrm{d}x - \int_0^1\frac{(1-x)^{n+1}}{2x-1}f(x)\,\mathrm{d}x\\
&= (n+1)\int_0^1\frac{x^{n+1}}{2x-1}f(x)\,\mathrm{d}x - \int_0^1\frac{x^{n+1}}{1-2x}f(1-x)\,\mathrm{d}x\\
&= (n+1)\int_0^1 x^n\frac{x}{2x-1}\bigl(f(x)+f(1-x)\bigr)\,\mathrm{d}x\,,
\end{aligned}$$

where in the second integral we substituted $1-x$ for x.

Now we can see that above transformations are possible iff all integrals exists, i.e. function $\frac{1}{2x-1}f(x)$ is integrable. Even in the opposite case we can eliminate the integrability problem by equating the function to zero in neighbourhood $1/2$, i.e. to proceed as follows.

Let $\varepsilon > 0$, $\tilde{f}(x) = \left\{ \begin{array}{cl} 0, & x \in (\frac{1}{4}, \frac{3}{4}) \\ f(x), & \text{otherwise} \end{array} \right.$ and let n_0 be such that

$$(n+1)^2 \left(\frac{3}{4}\right)^n < \frac{\varepsilon}{\max\limits_{x \in [\frac{1}{4}, \frac{3}{4}]} |f(x)|}$$

for all $n \geq n_0$. Then we have

$$\begin{aligned}
&\left| (n+1) \sum_{k=0}^{n} \int_0^1 x^k (1-x)^{n-k} f(x) \,\mathrm{d}x - (n+1) \sum_{k=0}^{n} \int_0^1 x^k (1-x)^{n-k} \tilde{f}(x) \,\mathrm{d}x \right| \\
&\qquad \leq (n+1) \sum_{k=0}^{n} \int_{\frac{1}{4}}^{\frac{3}{4}} x^k (1-x)^{n-k} \big|f(x)\big| \,\mathrm{d}x < (n+1)^2 \left(\frac{3}{4}\right)^n \cdot \max_{x \in [\frac{1}{4}, \frac{3}{4}]} |f(x)| = \varepsilon
\end{aligned}$$

for all $n \geq n_0$. Hence

$$\begin{aligned}
\lim_{n \to \infty} (n+1) \sum_{k=0}^{n} \int_0^1 x^k (1-x)^{n-k} f(x) \,\mathrm{d}x &= \lim_{n \to \infty} (n+1) \sum_{k=0}^{n} \int_0^1 x^k (1-x)^{n-k} \tilde{f}(x) \,\mathrm{d}x \\
&= \lim_{n \to \infty} (n+1) \int_0^1 x^n \frac{x}{2x-1} \big(\tilde{f}(x) + \tilde{f}(1-x)\big) \,\mathrm{d}x \\
&= \tilde{f}(1) + \tilde{f}(0) = f(0) + f(1).
\end{aligned}$$

$\square$

The 30th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 2nd April 2022
Category I

Problem 4 *In a box there are* 31, 41 *and* 59 *stones coloured, respectively, red, green and blue. Three players, having t-shirts of these three colours, play the following game. They sequentially make one of two moves:*

(I) either remove three stones of one colour from the box,

(II) or replace two stones of different colours by two stones of the third colour.

The game ends when all the stones in the box have the same colour and the winner is the player whose t-shirt has this colour. Assuming that the players play optimally, is it possible to decide whether the game ends and who will win, depending on who the starting player is? [Leszek Pieniążek / Jagiellonian University]

Solution We will show that if red player starts the game and the players use an optimal strategy the game will never stop. On the other hand, if any other player starts, red will win.

Let a_n, b_n, c_n be the numbers of stones of the three colours after n moves and define two functions:

$$N(n) = a_n + b_n + c_n \pmod 3 \qquad \text{and} \qquad D(n) = a_n + 2b_n \pmod 3.$$

It is easy to check that $N(n)$ and $D(n)$ are constant regardless of the moves made by players. $N(0) = 2$ and $D(0) = 2$, which implies that if the game ends after n moves, then $a_n \equiv 2 \pmod 3$ and $b_n = c_n = 0$ (easy check: two of the numbers should be 0 and the last one is given by $N(n) = D(n) = 2$), so the only player who can win is red. Thus

red wins iff game stops, the other players try to play infinitely.

We will prove that red has a strategy to win unless he starts.

At first stage, as long as it is possible, red removes three stones of any colour (or ends the game if it is possible – we already proved that he is the winner in this situation). Number of stones on the pile strictly decreases after his move, so at some time there will be 2 or 5 stones ($N(n) = 2$ and if there are more stones, red player can remove 3 stones). Now the only possible (unordered) combinations of numbers a_n, b_n, c_n can be seen on the diagram below. Arrows show how they can change in allowed moves.

$$\begin{tikzcd}
(011) \arrow[r] & (002) & (005) \\
(014) \arrow[u] \arrow[r] & (023) \arrow[u] \arrow[d] & (113) \arrow[l] \arrow[u] \arrow[ull] \\
& (122) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Red can win in one move for vertices 011, 023, 113. From 014 red goes to the winning vertex 011. The only possibility for draw is when red player moves from vertex 122. The other two players can go back to 122 in their moves and the game is infinite. Thus we have to determine who will have his move in vertex 122.

All possible (ordered!) values of $a_n, b_n, c_n \pmod 3$ are shown on the following diagram (easy check). Starting vertex corresponding to $(31, 41, 59)$ is 122. Arrows show changes according to moves of type (II). Moves (I) do not change the state of the diagram.

$$\begin{tikzcd}
(011) \arrow[r] & (200) \arrow[d] \\
& (122) \arrow[ul]
\end{tikzcd}$$

We analyse what happens if we obtain $(1, 2, 2)$. This means that the players made 42 moves of type (I) (as 126 stones where removed) and $3k$ moves (II) (k is the number of loops made on the graph), thus total number of moves is divisible by 3. So it is again the starting player's turn. If he is red, the other players can continue the game without end. If any other player started, red will be the winner. $\square$