The 30th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 2nd April 2022
Category II

Problem 1 *Determine whether there exists a differentiable function* $f\colon [0,1] \to \mathbb{R}$ *such that*

$$f(0) = f(1) = 1\,, \quad |f'(x)| \le 2 \text{ for all } x \in [0,1] \quad \text{and} \quad \left| \int_0^1 f(x)\,\mathrm{d}x \right| \le \frac{1}{2}\,.$$

[10 points]

Problem 2 *For any given pair of positive integers* $m > n$ *find all* $a \in \mathbb{R}$ *for which the polynomial* $x^m - ax^n + 1$ *can be expressed as a quotient of two nonzero polynomials with real nonnegative coefficients.* [10 points]

Problem 3 *Let* $x_1, \dots, x_n$ *be given real numbers with* $0 < m \le x_i \le M$ *for each* $i \in \{1, \dots, n\}$. *Let* X *be the discrete random variable uniformly distributed on* $\{x_1, \dots, x_n\}$. *The mean* μ *and the variance* σ^2 *of* X *are defined as*

$$\mu(X) = \frac{x_1 + \cdots + x_n}{n} \quad \text{and} \quad \sigma^2(X) = \frac{(x_1 - \mu(X))^2 + \cdots + (x_n - \mu(X))^2}{n}\,.$$

By X^2 *denote the discrete random variable uniformly distributed on* $\{x_1^2, \dots, x_n^2\}$. *Prove that*

$$\sigma^2(X) \ge \left(\frac{m}{2M^2}\right)^2 \sigma^2(X^2)\,.$$

[10 points]

Problem 4 *A function* $f\colon \mathbb{Z}^+ \to \mathbb{R}$ *is called* multiplicative *if for every* $a, b \in \mathbb{Z}^+$ *with* $\gcd(a,b) = 1$ *we have* $f(ab) = f(a)f(b)$. *Let* g *be the multiplicative function given by*

$$g(p^\alpha) = \alpha p^{\alpha - 1}\,,$$

where $\alpha \in \mathbb{Z}^+$ *and* $p > 0$ *is a prime. Prove that there exist infinitely many positive integers* n *such that*

$$g(n) + 1 = g(n+1)\,.$$

[10 points]