The 30$^{\text{th}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 2$^{\text{nd}}$ April 2022
Category I

Problem 1 *Assume that a real polynomial* $P(x)$ *has no real roots. Prove that the polynomial*

$$Q(x) = P(x) + \frac{P''(x)}{2!} + \frac{P^{(4)}(x)}{4!} + \dots$$

also has no real roots. [10 points]

Problem 2 *Let* $n \geq 1$. *Assume that* A *is a real* $n \times n$ *matrix which satisfies the equality*

$$A^7 + A^5 + A^3 + A - I = 0\,.$$

Show that $\det(A) > 0$. [10 points]

Problem 3 *Let* $f\colon [0,1] \to \mathbb{R}$ *be a given continuous function. Find the limit*

$$\lim_{n\to\infty} (n+1) \sum_{k=0}^{n} \int_0^1 x^k (1-x)^{n-k} f(x)\, \mathrm{d}x\,.$$

[10 points]

Problem 4 *In a box there are 31, 41 and 59 stones coloured, respectively, red, green and blue. Three players, having t-shirts of these three colours, play the following game. They sequentially make one of two moves:*

(I) either remove three stones of one colour from the box,

(II) or replace two stones of different colours by two stones of the third colour.

The game ends when all the stones in the box have the same colour and the winner is the player whose t-shirt has this colour. Assuming that the players play optimally, is it possible to decide whether the game ends and who will win, depending on who the starting player is? [10 points]