The 29th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 29th March 2019
Category II

Problem 1

a) *Is it true that for every non-empty set A and every associative operation $*\colon A \times A \to A$ the conditions*

$$x * x * y = y \qquad \text{and} \qquad y * x * x = y \qquad \text{for every } x, y \in A$$

imply commutativity of $*$?

b) *Is it true that for every non-empty set A and every associative operation $*\colon A \times A \to A$ the condition*

$$x * x * y = y \qquad \text{for every } x, y \in A$$

implies commutativity of $*$?

[10 points]

Problem 2 *Find all twice differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ such that*

$$f''(x)\cos(f(x)) \geq (f'(x))^2 \sin(f(x)) \qquad \text{for every } x \in \mathbb{R}\,. \tag{1}$$

[10 points]

Problem 3 *Let p be an even non-negative continuous function with $\int_{\mathbb{R}} p(x)\,\mathrm{d}x = 1$ and let n be a positive integer. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent identically distributed random variables with density function p. Define*

$$\begin{aligned} X_0 &= 0\,, \\ X_1 &= X_0 + \xi_1\,, \\ X_2 &= X_1 + \xi_2\,, \\ &\vdots \\ X_n &= X_{n-1} + \xi_n\,. \end{aligned}$$

Prove that the probability that all the random variables $X_1, X_2, \ldots, X_{n-1}$ lie between X_0 and X_n equals $\frac{1}{n}$.

[10 points]

Problem 4 *Let $D = \{z \in \mathbb{C} : \operatorname{Im} z > 0, \operatorname{Re} z > 0\}$. Let $n \geq 1$ and let $a_1, \ldots, a_n \in D$ be distinct complex numbers. Define*

$$f(z) = z \cdot \prod_{j=1}^{n} \frac{z - a_j}{z - \overline{a_j}}\,.$$

Prove that f' has at least one root in D. [10 points]