The 28^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13^{th} April 2018
Category I

Problem 1 *Every point of the rectangle $R = [0,4] \times [0,40]$ is coloured using one of four colours. Show that there exist four points in R with the same colour that form a rectangle having integer side lengths.* [10 points]

Solution Assume, that the rectangle is $[0,4] \times [0,40]$. For any j there are at least two points of the same colour in $A_j = \{(j,k) : k = 0,1,2,3,4\}$. There are 4 colours and 10 two-element subsets of the set $\{0,1,2,3,4\}$, so there are $j_1, j_2 \in \{0,1,\dots 40\}$ such that the two corresponding points in A_{j_1} and A_{j_2} have the same colour. $\square$

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Problem 2 *Find all prime numbers p such that p^3 divides the determinant*

$$\begin{vmatrix} 2^2 & 1 & 1 & \cdots & 1 \\ 1 & 3^2 & 1 & \cdots & 1 \\ 1 & 1 & 4^2 & & 1 \\ \vdots & \vdots & & \ddots & \\ 1 & 1 & 1 & & (p+7)^2 \end{vmatrix}.$$

[10 points]

Solution The answer is $p \in \{2, 3, 5, 181\}$.

Let $n = p + 6$. Denote by D the determinant in the statement of the problem. Subtracting the first row from all the remaining rows, we get

$$\begin{vmatrix} 2^2 & 1 & 1 & & 1 \\ -3 & 3^2-1 & 0 & \cdots & 0 \\ -3 & 0 & 4^2-1 & & 0 \\ & \vdots & & \ddots & \vdots \\ -3 & 0 & 0 & \cdots & (p+7)^2-1 \end{vmatrix}.$$

Hence

$$D = 3 \cdot (3^2-1) \cdots ((n+1)^2-1) \cdot \begin{vmatrix} \frac{2^2}{3} & \frac{1}{3^2-1} & \frac{1}{4^2-1} & & \frac{1}{(n+1)^2-1} \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix}.$$

Adding all the columns, except for the first one, of the last determinant to the first column, we obtain

$$D = 3 \cdot \prod_{k=3}^{n+1} (k^2-1) \cdot \begin{vmatrix} \frac{2^2}{3} + \sum_{k=3}^{n+1} \frac{1}{k^2-1} & \frac{1}{3^2-1} & \frac{1}{4^2-1} & & \frac{1}{(n+1)^2-1} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$= 3 \cdot \prod_{k=3}^{n+1} (k^2-1) \cdot \left(\frac{2^2}{3} + \sum_{k=3}^{n+1} \frac{1}{k^2-1} \right) = \frac{7n^2+17n+8}{8} (n!)^2 .$$

One can easily see that the prime numbers $p = 2, 3, 5$ satisfy the condition of the problem.

Assume that $p > 6$. Then $p < n = p + 6 < 2p$, so that $(n!)^2$ is divisible by p^2, but not by p^3. Therefore p divides $7n^2 + 17n + 8$. Then $7n^2 + 17n + 8 \equiv 7 \cdot 6^2 + 17 \cdot 6 + 8 = 2 \cdot 181 \pmod{p}$. (Recall that $n = p + 6$.) Hence p divides 181. Since 181 is a prime number, we obtain that $p = 181$. $\square$

Problem 3 *Let n be a positive integer and let $x_1, \dots, x_n$ be positive real numbers satisfying $|x_i - x_j| \le 1$ for all pairs (i, j) with $1 \le i < j \le n$. Prove that*

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \dots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \ge \frac{x_2+1}{x_1+1} + \frac{x_3+1}{x_2+1} + \dots + \frac{x_n+1}{x_{n-1}+1} + \frac{x_1+1}{x_n+1}.$$

[10 points]

Solution Denote $h(t) = t - \log t$. The idea is to prove

$$h\left(\frac{x}{y}\right) \ge h\left(\frac{y+1}{x+1}\right)$$

whenever $x, y > 0$ and $|x-y| \le 1$. Then summing up these inequalities for $(x, y) = (x_i, x_{i+1})$ we get the desired inequality, since the logarithms cancel out. Note that $h'(t) = 1 - 1/t$, so h decreases on $(0, 1]$ and increases on $[1, +\infty)$. We use two simple inequalities:

1. $h(t) \ge h(1/t)$ for $t \ge 1$. Indeed, denoting $1/t = 1-s$, $s \in [0, 1)$, it rewrites as $1/(1-s)-1+s \ge -2\log(1-s)$, that follows from expanding both parts as series in s.
2. $h(1-s) \ge h(1+s)$ for $s \in [0, 1)$. This rewrites as $-\log(1-s) + \log(1+s) \ge 2s$, that follows from expanding as series in s.

Now if $x \ge y$, we have $h(\frac{x}{y}) \ge h(\frac{y}{x}) \ge h(\frac{y+1}{x+1})$, the second inequality follows from monotonicity of h on $(0, 1]$ and obvious inequality $\frac{y}{x} \le \frac{y+1}{x+1}$. Note that here we did not use that $|x - y| \le 1$.

If $x < y \le x + 1$, we get

$$h\Big(\frac{x}{y}\Big) = h\Big(1 - \frac{y-x}{y}\Big) \ge h\Big(1 + \frac{y-x}{y}\Big) \ge h\Big(1 + \frac{y-x}{x+1}\Big) = h\Big(\frac{y+1}{x+1}\Big)$$

as desired. $\square$

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Problem 4 *Determine all possible (finite or infinite) values of*

$$\lim_{x\to-\infty} f(x) - \lim_{x\to+\infty} f(x)\,,$$

if $f\colon \mathbb{R}\to\mathbb{R}$ is a strictly decreasing continuous function satisfying

$$f(f(x))^4 - f(f(x)) + f(x) = 1$$

for all $x\in\mathbb{R}$. [10 points]

Solution Obviously, the difference of the limits must be positive. We first show that it must be smaller than $(1+z)^4-1$ where $z=\dfrac{3}{4\sqrt[3]{4}}$. Then we show that all values in $(0,(1+z)^4-1)$ can be attained.

Assume that f satisfies all the desired properties. Let us denote $b=\lim\limits_{x\to-\infty} f(x)$ and $a=\lim\limits_{x\to+\infty} f(x)$. Then $b-a>0$ and $H_f=(a,b)$ is the range of f. For all $y\in H_f$ we have $g(f(y))=1-y$, where $g(x)=x^4-x$. Function g attains its minimal value $-z$ at $x=\dfrac{1}{\sqrt[3]{4}}$. So, $g(\mathbb{R})=[-z,+\infty)$. It follows that for every $y\in H_f$, $1-y\in[-z,+\infty)$, i.e. $H_f\subset(-\infty,1+z]$. So, we have $a<b\le 1+z$.

Function g is decreasing on $I_1=\big(-\infty,\frac{1}{\sqrt[3]{4}}\big]$ and increasing on $I_2=\big[\frac{1}{\sqrt[3]{4}},+\infty\big)$. Let us denote $g_1^{-1}=(g|_{I_1})^{-1}$ and $g_2^{-1}=(g|_{I_2})^{-1}$. Since f is continuous, we have either $f(y)=g_1^{-1}(1-y)$ on H_f or $f(y)=g_2^{-1}(1-y)$ on H_f. However, since f is decreasing, we have $f(y)=g_2^{-1}(1-y)$ on H_f. Let us denote $h(x)=g_2^{-1}(1-x)$, $x\in(-\infty,1+z]$. Note that h is decreasing, $h(1)=1$, and $h^{-1}(x)=1+x-x^4$. Since $f(y)=h(y)$ on (a,b) and $f(y)<b\le 1+z$ on $\mathbb{R}$, we have for all $\varepsilon>0$ small enough $h(a+\varepsilon)=f(a+\varepsilon)<1+z$, i.e. $h(a)\le 1+z$. Obviously, by continuity we have $h(a)=f(a)$, so $h(a)=b$ leads to contradiction with $b\notin H_f$, e.g. we obtain $h(a)<1+z$. Applying g to both sides of this inequality we get

$$1-a<g\,(1+z)=(1+z)^4-(1+z)\,,\quad \text{i.e.}\quad a>2+z-(1+z)^4\,.$$

Since $b\le 1+z$, it follows that $0<b-a<(1+z)^4-1$.

We show that for any number $c\in(0,(1+z)^4-1)$ there exists a suitable function f. In fact, it is sufficient to find a and b with $b-a=c$ such that $h(a)<b$, $h(b)>a$; then we define $f(x)=h(x)$ on $[a,b]$ and on $(-\infty,a)$ and $(b,+\infty)$ we take any decreasing continuous function with $\lim\limits_{x\to-\infty} f(x)=b$ and $\lim\limits_{x\to+\infty} f(x)=a$ and with appropriate limits at a and b (to be continuous on $\mathbb{R}$).

We show in the Lemma below that $h^{-1}(b)<h(b)$ for every $b\in J=(1,1+z)$. Let us define for each $b\in J$

$$a(b)=\lambda(b)h^{-1}(b)+(1-\lambda(b))h(b)\,,\quad \text{where}\quad \lambda(b)=\frac{b-1}{z}\,.$$

Then $a(b)$ is a convex combination of $h^{-1}(b)$ and $h(b)$, hence $a(b)<h(b)$ and $a(b)>h^{-1}(b)$, i.e. $h(a(b))<b$. Moreover, function $b\mapsto b-a(b)$ is continuous on J with

$$\lim_{b\to1+} b-a(b)=1-h(1)=0\quad\text{and}\quad \lim_{b\to1+z-} b-a(b)=1+z-h^{-1}(1+z)=1+z-(1+(1+z)-(1+z)^4)=(1+z)^4-1,$$

so $b-a(b)$ attains all values from $\big(0,(1+z)^4-1\big)$. To complete the solution it only remains to prove the following lemma.

Lemma *$h^{-1}(b)<h(b)$ holds for all $b\in(1,1+z)=J$.*

Proof First, $h(b)>h(1+z)=\dfrac{1}{\sqrt[3]{4}}$ on J. So, it is sufficient to prove the inequality for all b satisfying $h^{-1}(b)>\dfrac{1}{\sqrt[3]{4}}$, i.e.

$$1+b-b^4>\frac{1}{\sqrt[3]{4}}. \tag{1}$$

For such b we can apply h^{-1} to both sides of the inequality $h^{-1}(b)<h(b)$, i.e. we only need to prove $h^{-1}(h^{-1}(b))>b$. Since

$$h^{-1}(h^{-1}(b))=h^{-1}(1+b-b^4)=1+(1+b-b^4)-(1+b-b^4)^4,$$

we need to show that $\phi(b) = 1 + (1 + b - b^4) - (1 + b - b^4)^4 - b > 0$ on J. Obviously, $\phi(1) = 0$, so it is sufficient to show $\phi' > 0$ on the subinterval of J where (1) holds. We have (by (1) and $b > 1$)

$$\begin{aligned}\phi'(b) &= 1 - 4b^3 - 4(1 + b - b^4)^3(1 - 4b^3) - 1 = 4\left((1 + b - b^4)(4b^3 - 1) - b^3\right) > 4\left(\frac{1}{\sqrt[3]{4}}(4b^3 - 1) - b^3\right) \\ &= 4\left(b^3(4^{2/3} - 1) - 4^{-1/3}\right) > 4\left(4^{2/3} - 1 - 4^{-1/3}\right) = 4^{2/3}\left(4 - 4^{1/3} - 1\right) > 0.\end{aligned}$$

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