The 28th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13th April 2018
Category II

Problem 1 *Find all real solutions of the equation*

$$17^x + 2^x = 11^x + 2^{3x}\,.$$

[10 points]

Problem 2 *Let n be a positive integer and let $a_1 \le a_2 \le \cdots \le a_n$ be real numbers such that*

$$a_1 + 2a_2 + \cdots + na_n = 0\,.$$

Prove that

$$a_1[x] + a_2[2x] + \cdots + a_n[nx] \ge 0$$

for every real number x. (Here $[t]$ denotes the integer satisfying $[t] \le t < [t] + 1$.) [10 points]

Problem 3 *In $\mathbb{R}^3$ some n points are coloured. In every step, if four coloured points lie on the same line, Vojtěch can colour any other point on this line. He observes that he can colour any point $P \in \mathbb{R}^3$ in a finite number of steps (possibly depending on P). Find the minimal value of n for which this could happen.* [10 points]

Problem 4 *Compute the integral*

$$\iint\limits_{\mathbb{R}^2} \left(\frac{1 - \mathrm{e}^{-xy}}{xy} \right)^2 \mathrm{e}^{-x^2 - y^2}\, \mathrm{d}x\, \mathrm{d}y\,.$$

[10 points]