The 27th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 31st March 2017
Category II

Problem 1 *Let $(a_n)_{n=1}^{\infty}$ be a sequence with $a_n \in \{0,1\}$ for every n. Let $F\colon (-1,1) \to \mathbb{R}$ be defined by*

$$F(x) = \sum_{n=1}^{\infty} a_n x^n$$

and assume that $F\left(\frac{1}{2}\right)$ is rational. Show that F is the quotient of two polynomials with integer coefficients.

[10 points]

Solution $F(1/2)$ is the base 2 expansion of some real number in the unit interval. By assumption, it is rational and hence periodic. This implies the existence of some p, n with

$$a_{i+p} = a_i \text{ for all } i > n\,.$$

We may thus write

$$\begin{aligned} F(X) &= a_0 + a_1 X + \cdots + a_n X^n + (a_{n+1}X^{n+1} + \cdots + a_{n+p}X^{n+p}) \sum_{i=0}^{\infty} X^{ip} \\ &= a_0 + a_1 X + \cdots + a_n X^n + \frac{a_{n+1}X^{n+1} + \cdots + a_{n+p}X^{n+p}}{1 - X^p} \end{aligned}$$

from which the assertion follows. $\square$

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Problem 2 *Prove or disprove the following statement. If $g\colon (0,1) \to (0,1)$ is an increasing function and satisfies $g(x) > x$ for all $x \in (0,1)$, then there exists a continuous function $f\colon (0,1) \to \mathbb{R}$ satisfying $f(x) < f(g(x))$ for all $x \in (0,1)$, but f is not an increasing function.* [10 points]

Solution The statement is true, here is an example of such f: Let us denote $z := g(1/2)$. Then $z > \frac{1}{2}$. Let us define $f(x) := x$ for $x \in (0, \frac{1}{2}) \cup (z, 1)$ and we define $f := h$ on $[\frac{1}{2}, z]$ where $h : [\frac{1}{2}, z] \to [\frac{1}{2}, z]$ is any continuous function which is not increasing and satisfies $h(1/2) = 1/2$, $h(z) = z$ and $h(x) \in (1/2, z)$ on $(1/2, z)$. Then f is continuous on $(0,1)$, it is not increasing and it satisfies $f(x) < f(g(x))$. In fact, if $x \geq z$ or $x \in (0, 1/2)$ with $g(x) < 1/2$ we have

$$f(g(x)) = g(x) > x = f(x)\,.$$

If $x \in (0, 1/2)$ with $g(x) \geq 1/2$ we have $g(x) < z$,

$$f(g(x)) = h(g(x)) \geq 1/2 > x = f(x)\,.$$

If $x \in [1/2, z)$, then $g(x) \geq g(1/2) = z$,

$$f(g(x)) = g(x) \geq z > h(x) = f(x)\,.$$

$\square$

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Problem 3 *Let $n \geq 2$ be an integer. Consider the system of equations*

$$x_1 + \frac{2}{x_2} = x_2 + \frac{2}{x_3} = \ldots = x_n + \frac{2}{x_1}. \tag{1}$$

1. *Prove that* (1) *has infinitely many real solutions $(x_1, \ldots, x_n)$ such that the numbers $x_1, \ldots, x_n$ are distinct.*
2. *Prove that every solution $(x_1, \ldots, x_n)$ of* (1)*, such that the numbers $x_1, \ldots, x_n$ are not all equal, satisfies $|x_1 x_2 \cdots x_n| = 2^{n/2}$.*

[10 points]

Solution 1 (a) The main idea is to put $x_k = a \cdot \tan\left(t_0 + \dfrac{k\pi}{n}\right) + b$ with some real numbers a, b, t_0. It suffices to establish an identity like

$$\left(a \cdot \tan\left(t - \frac{\pi}{n}\right)t + b\right) + \frac{2}{a \cdot \tan t + b} = (*) = \text{const.}$$

Put $T = \tan t$ and $c = \tan \dfrac{\pi}{n}$; then

$$(*) = b + a\frac{T - c}{1 + cT} + \frac{2}{aT + b}.$$

Obviously we need $a = bc$ for the common denominator; then

$$(*) = b + bc\frac{T - c}{1 + cT} + \frac{2}{bcT + b} = 2b + \frac{2 - b^2(1 + c^2)}{b(1 + cT)}.$$

This expression is constant in T if and only if $b = \pm\sqrt{\dfrac{2}{1 + c^2}} = \pm\sqrt{2}\cos\dfrac{\pi}{n}$.

Hence, with the choice $b = \sqrt{2}\cos\dfrac{\pi}{n}$, $a = bc = \sqrt{2}\sin\dfrac{\pi}{n}$ and $x_k = a \cdot \tan\left(t_0 + \dfrac{k\pi}{n}\right) + b$ with some $t_0 \in [0, \pi)$, we achieve (1), except for finitely many t_0 when one of the tangents is undefined.

(b) Due to the cyclic symmetry we may assume $x_1 \neq x_2$. Then

$$x_2 - x_1 = \frac{2}{x_2} - \frac{2}{x_3} = \frac{2}{x_2 x_3} \cdot (x_3 - x_2) = \frac{2}{x_2 x_3} \cdot \frac{2}{x_3 x_4} \cdot (x_4 - x_3) = \ldots = \frac{2}{x_2 x_3} \cdot \frac{2}{x_3 x_4} \cdots \frac{2}{x_1 x_2}(x_2 - x_1),$$
$$x_1^2 x_2^2 \ldots x_n^2 = 2^n.$$

□

Solution 2 (b) Assume that

$$x_1 + \frac{2}{x_2} = x_2 + \frac{2}{x_3} = \ldots = x_n + \frac{2}{x_1} = A$$

with some real A and let $M = \begin{pmatrix} A & -2 \\ 1 & 0 \end{pmatrix}$. Then we have

$$M\begin{pmatrix} x_{k+1} \\ 1 \end{pmatrix} = \begin{pmatrix} Ax_{k+1} - 2 \\ x_{k+1} \end{pmatrix} = x_{k+1} \cdot \begin{pmatrix} A - \frac{2}{x_{k+1}} \\ 1 \end{pmatrix} = x_{k+1} \cdot \begin{pmatrix} x_k \\ 1 \end{pmatrix}.$$

By applying this for each k, we get

$$M^n\begin{pmatrix} x_k \\ 1 \end{pmatrix} = x_1 x_2 \cdots x_n \cdot \begin{pmatrix} x_k \\ 1 \end{pmatrix}$$

so, $\begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ 1 \end{pmatrix}$ are all eigenvectors of the matrix M^n with the common eigenvalue $x_1 x_2 \cdots x_n$. Since the numbers $x_1, \ldots, x_n$ are not all equal, the vectors $\begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ 1 \end{pmatrix}$ span the 2-dimensional space. So, M^n must be diagonal and $M^n = x_1 x_2 \cdots x_n \cdot I$.

From $\det M = 2$ we get $2^n = \det M^n = \det(x_1 x_2 \cdots x_n \cdot I) = (x_1 x_2 \cdots x_n)^2$, so $|x_1 x_2 \cdots x_n| = 2^{n/2}$.

(a) Choose the number A in such a way that the eigenvalues of M are $\sqrt{2} \cdot e^{\pm \frac{\pi i}{n}}$. That can be achieved by choosing $A = \operatorname{tr} M = -2\sqrt{2}\cos\dfrac{\pi}{n}$. Then, we achieve $M^n = 2^{n/2} I$ as well. Starting from an arbitrary vector $\begin{pmatrix} x_n \\ 1 \end{pmatrix}$, we can determine $\begin{pmatrix} x_{n-1} \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} x_1 \\ 1 \end{pmatrix}$ one by one; at the end the cycle will be closed. There are only finitely many starting values x_n when some x_k becomes zero by accident. $\square$

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Problem 4 *A positive integer t is called a Jane's integer if $t = x^3 + y^2$ for some positive integers x and y. Prove that for every integer $n \geq 2$ there exist infinitely many positive integers m such that the set of n^2 consecutive integers $\{m+1, m+2, \ldots, m+n^2\}$ contains exactly $n+1$ Jane's integers.* [10 points]

Solution Fix $n \geq 2$. Let throughout $C(m)$ denote the number of Czech integers in the set $\{m+1, m+2, \ldots, m+n^2\}$. With this notation, we need to show that $C(m) = n+1$ for infinitely many $m \in \mathbb{N}$.

Below, we will prove that there exist two infinite sequences of positive integers $S = \{s_1 < s_2 < s_3 < \ldots\}$ and $L = \{l_1 < l_2 < l_3 < \ldots\}$ such that $C(s_i) = 0$ and $C(l_i) \geq n+1$ for $i = 1, 2, 3, \ldots$. Then, for any $s \in S$, let us take the smallest $l \in L$ satisfying $l > s$. By the definitions of S and L, the list of nonnegative integers $C(s), C(s+1), \ldots, C(l)$ starts with the number $C(s) = 0$ and ends up with the number $g = C(l) \geq n+1$. Since $C(j+1) - C(j) \in \{-1, 0, 1\}$, the list $C(s), C(s+1), \ldots, C(l)$ contains every integer between 0 and g. In particular, it contains the integer $n+1$. Hence, $n+1 = C(m)$ for some m in the range $s \leq m \leq l$. Since one can choose infinitely many disjoint intervals $[s, l]$ as above, this would finish the proof.

Let us show the existence of the sequence S. Suppose that for an even positive integer u each of the intervals $[uk/2+1, u(k+1)/2]$, where $k = 1, 2, \ldots, 2u^5 - 1$, contains at least one Czech integer. Then, the interval $[u/2+1, u^6]$ contains at least $2u^5 - 1$ Czech integers. However, the interval $[1, u^6]$ contains at most u^5 Czech integers, since $t = x^3 + y^2 \leq u^6$ implies $1 \leq x \leq u^2$ and $1 \leq y \leq u^3$. Thus, $2u^5 - 1 \leq u^5$, which is impossible. Hence, at least one of the intervals $[uk/2+1, u(k+1)/2]$ is free of Czech integers. For this particular k, we have $C(uk/2) = 0$ if $u/2 \geq n^2$, so the element $s = uk/2$ for our sequence S can be selected in each interval $[u/2, u^6 - u/2]$, where $u \geq 2n^2$ is even.

It remains to show the existence of the sequence L. For $n \geq 3$, we can simply take $L = \{1^6, 2^6, 3^6, \ldots\}$. Then, the set $\{i^6+1, i^6+2, \ldots, i^6+n^2\}$ contains n Czech integers $(i^2)^3 + 1^2, (i^2)^3 + 2^2, \ldots, (i^2)^3 + n^2$ and one more Czech integer $2^3 + (i^3)^2$, since 2^3 is not a square and $2^3 < n^2$. Consequently, $C(i^6) \geq n+1$ for $i \in \mathbb{N}$, and so the proof (for each $n \geq 3$) is completed.

For $n = 2$, we will construct the sequence $L = \{v_1^3, v_2^3, v_3^3, \ldots\}$ with some positive integers $v_1 < v_2 < v_3 < \ldots$ satisfying $C(v_i^3) \geq 3$. Clearly, the set $\{v_i^3+1, v_i^3+2, v_i^3+3, v_i^3+4\}$ contains two Czech integers v_i^3+1 and v_i^3+4. In addition, v_i^3+2 is a Czech integer if, say, $v_i^3 + 2 = (v_i - 1)^3 + y^2$ for $y \in \mathbb{N}$ (and $v_i > 1$). This equality can be rewritten in the equivalent form

$$3(2v_i - 1)^2 + 9 = (2y)^2.$$

Now, since the fundamental solution of the Pell equation $X^2 - 3Y^2 = 1$ is $(X, Y) = (2, 1)$, and its odd powers $(2+\sqrt{3})^{2i-1}$ give infinitely many pairs $(X_i, Y_i) \in \mathbb{N}^2$, where $X_1 = 2 < X_2 < X_3 < \ldots$ are even, $Y_1 = 1 < Y_2 < Y_3 < \ldots$ are odd, and $X_i^2 - 3Y_i^2 = 1$, we can select $v_i = (3Y_i+1)/2 > 1$ and $y = 3X_i/2$. With this choice, it is clear that

$$3(2v_i - 1)^2 + 9 = 3(3Y_i)^2 + 9 = 9(3Y_i^2 + 1) = 9X_i^2 = (2y)^2,$$

as claimed. This completes the proof for $n = 2$. $\square$