The 27^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 31^{st} March 2017
Category I

Problem 1 *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying*

$$f(x+2y) = 2f(x)f(y)$$

for every $x, y \in \mathbb{R}$*. Prove that* f *is constant.* [10 points]

Solution By taking $y = 0$ we obtain $f(x) = 2f(x)f(0)$ for any $x \in \mathbb{R}$. If $f(0) \neq \frac{1}{2}$, it follows that $f(x) = 0$ for all $x \in \mathbb{R}$ and f is a constant function.

Therefore, in the rest of the proof assume that $f(0) = \frac{1}{2}$. After taking $x = 0$ in $f(x+2y) = 2f(x)f(y)$ we get $f(2y) = f(y)$ for any $y \in \mathbb{R}$.

Let $a \in \mathbb{R}$. One can easily check that $f(a) = f(\frac{a}{2^n})$ for all $n \in \mathbb{N}$. Therefore, since f is continuous, we obtain that

$$f(a) = \lim_{n\to\infty} f(a) = \lim_{n\to\infty} f\left(\frac{a}{2^n}\right) = f\left(\lim_{n\to\infty} \frac{a}{2^n}\right) = f(0) = \frac{1}{2}$$

and the proof is complete. $\square$

Problem 2 *We say that we extend a finite sequence of positive integers* $(a_1, \ldots, a_n)$ *if we replace it by*

$$(1, 2, \ldots, a_1 - 1, a_1, 1, 2, \ldots, a_2 - 1, a_2, 1, 2, \ldots, a_3 - 1, a_3, \ldots, 1, 2, \ldots, a_n - 1, a_n),$$

i.e., each element k *of the original sequence is replaced by* $1, 2, \ldots, k-1, k$*. Géza takes the sequence* $(1, 2, \ldots, 9)$ *and he extends it* 2017 *times. Then he chooses randomly one element of the resulting sequence. What is the probability that the chosen element is* 1*?* [10 points]

Solution We show by induction that number $j \in \{1, 2, \ldots, 9\}$ occurs $\binom{9-j+m}{m}$ times in the m times extended sequence. Every $j \in \{1, 2, \ldots, 9\}$ occurs once in the original (0 times extended) sequence. Let us now perform the induction step. Number of occurences of j in the m times extended sequence is equal to the number of occurences of all numbers larger than or equal to j in $(m-1)$ times extended sequence, which is (by the inductive assumption) equal to

$$\binom{9-j+m-1}{m-1} + \binom{9-(j+1)+m-1}{m-1} + \cdots + \binom{9-9+m-1}{m-1}.$$

This sum is by a well known identity equal to $\binom{9-j+m}{m}$ and the induction step is done. It follows that in the 2017 times extended sequence number 1 occurs $\binom{9-1+2017}{2017}$ times among

$$\sum_{j=1}^{9} \binom{9-j+2017}{2017} = \binom{9+2017}{2018}$$

elements. So, the probability of choosing 1 is

$$\frac{2025!}{2017!8!} \cdot \frac{2018!8!}{2026!} = \frac{2018}{2026}.$$

□

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Problem 3 *Let P be a convex polyhedron. Jaroslav writes a non-negative real number to every vertex of P in such a way that the sum of these numbers is 1. Afterwards, to every edge he writes the product of the numbers at the two endpoints of that edge. Prove that the sum of the numbers at the edges is at most $\frac{3}{8}$.* [10 points]

First solution We consider the graph $G = (V, E)$ on the sphere (or equivalently, in the plane) induced by the vertices and edges of P and write $v \sim w$ if v, w are connected by an edge. Denote by

$$A = \left\{ (a_v)_{v \in V} \in \mathbb{R}^{|V|} \colon a_v \geq 0, \sum_{v \in V} a_v = 1 \right\}$$

the set of possible numberings, and let

$$f\left((a_v)_{v \in V}\right) = \sum_{v \sim w} a_v a_w$$

We need to prove $f\left((a_v)_{v \in V}\right) \leq \frac{3}{8}$ whenever $(a_v)_{v \in V} \in A$.

A is compact and f is continuous, therefore $m := \max_{(a_v)_{v \in V} \in A} f\left((a_v)_{v \in V}\right)$ is assumed. Let $(a_v^*)_{v \in V} \in A$ be such that $f\left((a_v^*)_{v \in V}\right) = m$ and such that the number of zeroes in $(a_v^*)_{v \in V}$ is maximal among all configurations $(a_v)_{v \in V} \in A$ such that $f\left((a_v)_{v \in V}\right) = m$. We are going to study this particular optimal configuration.

Lemma 1 *Any two vertices w, w' with non-zero $a_w^*, a_{w'}^*$ are connected by an edge.*

Proof Let w, w' be two vertices with non-zero $a_w^*, a_{w'}^*$, and assume they are not connected by an edge. W.l.o.g assume $a_w^* \leq a_{w'}^*$. Note that

$$f\left((a_v^*)_{v \in V}\right) = a_w^* \sum_{v \sim w^*} a_v^* + a_{w'}^* \sum_{v \sim w'^*} a_v^* + \sum_{\substack{v, v' \in V \setminus \{w, w'\} \\ v \sim v'}} a_v^* a_{v'}^*$$

Now consider

$$g(\varepsilon) := f\left(a_w^* - \varepsilon, a_{w'}^* + \varepsilon, (a_v)_{v \in V \setminus \{w, w'\}}\right) .$$

This is a linear function of ε. Because $(a_v^*)_{v \in V}$ is an optimal configuration and $a_w^*, a_{w'}^*$ are nonzero, we have that

$$\left(a_w^* - \varepsilon, a_{w'}^* + \varepsilon, (a_v)_{v \in V \setminus \{w, w'\}}\right) \in A$$

for all ε in a neighborhood of 0 and

$$g(\varepsilon) = f\left(a_w^* - \varepsilon, a_{w'}^* + \varepsilon, (a_v)_{v \in V \setminus \{w, w'\}}\right) \leq f\left((a_v^*)_{v \in V}\right) = g(0)$$

But then g is a linear function with a local maximum, so it must be constant. This implies

$$m = g(0) = g(a_w^*) = f\left(0, a_w^* + a_{w'}^*, (a_v)_{v \in V \setminus w, w'}\right) .$$

On the other hand, $\left(0, a_w^* + a_{w'}^*, (a_v)_{v \in V \setminus \{w, w'\}}\right) \in A$. This implies that $\left(0, a_w^* + a_{w'}^*, (a_v)_{v \in V \setminus \{w, w'\}}\right)$ is an optimal configuration with an even larger number of zeroes than $(a_v^*)_{v \in V}$, contradicting the choice of $(a_v^*)_{v \in V}$.

□

The lemma implies that the vertices with non-zero a_v^* induce a complete subgraph of G. Because G is a planar graph, this subgraph must be planar as well. It is well-known that a complete graph is planar if and only if it has at most 4 vertices. We conclude that at most 4 of the a_v^* are non-zero. Let these be $a_{v_1}, a_{v_2}, a_{v_3}, a_{v_4}$ (some of them possibly 0, if there are less than 4 non-zero a_v^*). Then

$$m = f\left((a_v^*)_{v \in V}\right) \leq a_{v_1} a_{v_2} + a_{v_1} a_{v_3} + a_{v_1} a_{v_4} + a_{v_2} a_{v_3} + a_{v_2} a_{v_4} + a_{v_3} a_{v_4}$$

and by the following lemma we conclude $m \leq \frac{3}{8}$, as claimed.

Lemma 2 *Let $a, b, c, d \in \mathbb{R}$ such that $a, b, c, d \geq 0$, $a + b + c + d = 1$. Then*

$$ab + ac + ad + bc + bd + cd \leq \frac{3}{8}$$

Proof We have that

$$\begin{aligned}
\frac{3}{8} - ab + ac + ad + bc + bd + cd &= \frac{3}{8}(a+b+c+d)^2 - (ab+ac+ad+bc+bd+cd) \\
&= \frac{3}{8}\left(a^2+b^2+c^2+d^2\right) - \frac{1}{4}(ab+ac+ad+bc+bd+cd) \\
&= \frac{1}{8}\left((a-b)^2+(a-c)^2+(a-d)^2+(b-c)^2+(b-d)^2+(c-d)^2\right) \\
&\geq 0
\end{aligned}$$

$\square$

$\square$

Second solution Let G, A and f be as in the first solution. By the four color theorem applied to the planar graph G, one can partition V into four sets V_1, V_2, V_3, V_4 in such a way that there are no edges between vertices within the same set. Now for $(a_v)_{v \in V} \in A$ one can write

$$\begin{aligned}
f\left((a_v)_{v\in V}\right) &= \sum_{\substack{v,w \in V \\ v \sim w}} a_v a_w \\
&\leq \sum_{1 \leq i < j \leq 4} \sum_{\substack{v \in V_i, w \in V_j \\ v \sim w}} a_v a_w \\
&\leq \sum_{1 \leq i < j \leq 4} \sum_{v \in V_i, w \in V_j} a_v a_w \\
&= \sum_{1 \leq i < j \leq 4} \left(\sum_{v \in V_i} a_v\right)\left(\sum_{w \in V_j} a_w\right)
\end{aligned}$$

Now the conclusion follows by applying lemma 2 to $\sum_{v\in V_1} a_v, \sum_{v\in V_2} a_v, \sum_{v\in V_3} a_v, \sum_{v\in V_4} a_v$. $\square$

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Problem 4 *Let $f\colon (1,\infty) \to \mathbb{R}$ be a continuously differentiable function satisfying $f(x) \leq x^2 \log(x)$ and $f'(x) > 0$ for every $x \in (1,\infty)$. Prove that*

$$\int_1^\infty \frac{1}{f'(x)}\,\mathrm{d}x = \infty\,.$$

[10 points]

Solution Let $a > e$. By the Cauchy-Schwarz inequality,

$$\left(\int_e^a \frac{1}{f'(x)}dx\right)\left(\int_e^a \frac{f'(x)}{(x\log(x))^2}dx\right) \geq \left(\int_e^a \frac{1}{x\log(x)}dx\right)^2 = (\log(\log(a)))^2$$

On the other hand, by integration by parts and the assumption, we get

$$\begin{aligned}\int_e^a \frac{f'(x)}{(x\log(x))^2}dx &= \frac{f(x)}{(x\log(x))^2}\Bigg|_e^a + 2\int_e^a \frac{f(x)}{(x\log(x))^3}(\log(x)+1)dx \\ &= \frac{f(a)}{(a\log(a))^2} - \frac{f(e)}{e^2} + 2\int_e^a \frac{f(x)}{(x\log(x))^3}(\log(x)+1)dx \\ &\leq \frac{1}{\log(a)} - \frac{f(e)}{e^2} + 2\int_e^a \frac{x^2\log(x)}{(x\log(x))^3}(\log(x)+1)dx \\ &= \frac{1}{\log(a)} - \frac{f(e)}{e^2} + 2\int_e^a \frac{1}{x\log(x)} + \frac{1}{x\log(x)^2}dx \\ &= \frac{1}{\log(a)} - \frac{f(e)}{e^2} + 2\log(\log(a)) + 2 - 2\frac{1}{\log(a)} \\ &\leq 2\log(\log(a)) + 2 - \frac{f(e)}{e^2}\end{aligned}$$

Because f is strictly increasing, $\int_e^a \frac{f'(x)}{(x\log(x))^2}dx$ is positive. Therefore we can combine the two inequalities as follows:

$$\begin{aligned}\int_e^a \frac{1}{f'(x)}dx &\geq \frac{\left(\int_e^a \frac{1}{x\log(x)}dx\right)^2}{\int_e^a \frac{f'(x)}{(x\log(x))^2}dx} \\ &\geq \frac{(\log(\log(a)))^2}{2\log(\log(a)) + 2 - \frac{f(e)}{e^2}}\end{aligned}$$

The right hand side obviously tends to infinity as $a \to \infty$. Hence also

$$\int_1^\infty \frac{1}{f'(x)}dx \geq \int_e^\infty \frac{1}{f'(x)}dx = \lim_{a\to\infty}\int_e^a \frac{1}{f'(x)}dx = \infty$$

$\square$