The 27^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 31^{st} March 2017
Category I

Problem 1 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying*

$$f(x+2y) = 2f(x)f(y)$$

for every $x, y \in \mathbb{R}$. Prove that f is constant. [10 points]

Problem 2 *We say that we extend a finite sequence of positive integers $(a_1, \ldots, a_n)$ if we replace it by*

$$(1, 2, \ldots, a_1 - 1, a_1, 1, 2, \ldots, a_2 - 1, a_2, 1, 2, \ldots, a_3 - 1, a_3, \ldots, 1, 2, \ldots, a_n - 1, a_n)\,,$$

i.e., each element k of the original sequence is replaced by $1, 2, \ldots, k-1, k$. Géza takes the sequence $(1, 2, \ldots, 9)$ and he extends it 2017 times. Then he chooses randomly one element of the resulting sequence. What is the probability that the chosen element is 1? [10 points]

Problem 3 *Let P be a convex polyhedron. Jaroslav writes a non-negative real number to every vertex of P in such a way that the sum of these numbers is 1. Afterwards, to every edge he writes the product of the numbers at the two endpoints of that edge. Prove that the sum of the numbers at the edges is at most $\frac{3}{8}$.* [10 points]

Problem 4 *Let $f\colon (1, \infty) \to \mathbb{R}$ be a continuously differentiable function satisfying $f(x) \le x^2 \log(x)$ and $f'(x) > 0$ for every $x \in (1, \infty)$. Prove that*

$$\int_1^\infty \frac{1}{f'(x)}\,\mathrm{d}x = \infty\,.$$

[10 points]