The 25th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 27th March 2015
Category I

Problem 1 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$. Prove that there exists $x \in [0,1]$ such that*

$$\frac{4}{\pi}\big(f(1) - f(0)\big) = (1 + x^2) f'(x).$$

[10 points]

Problem 2 *Consider the infinite chessboard whose rows and columns are indexed by positive integers. Is it possible to put a single positive rational number into each cell of the chessboard so that each positive rational number appears exactly once and the sum of every row and of every column is finite?* [10 points]

Problem 3 *Let $P(x) = x^{2015} - 2x^{2014} + 1$ and $Q(x) = x^{2015} - 2x^{2014} - 1$. Determine for each of the polynomials P and Q whether it is a divisor of some nonzero polynomial $c_0 + c_1 x + \cdots + c_n x^n$ whose coefficients c_i are all in the set $\{1, -1\}$.* [10 points]

Problem 4 *Let m be a positive integer and let p be a prime divisor of m. Suppose that the complex polynomial $a_0 + a_1 x + \ldots + a_n x^n$ with $n < \frac{p}{p-1}\varphi(m)$ and $a_n \neq 0$ is divisible by the cyclotomic polynomial $\Phi_m(x)$. Prove that there are at least p nonzero coefficients a_i.*

The cyclotomic polynomial $\Phi_m(x)$ is the monic polynomial whose roots are the m-th primitive complex roots of unity. Euler's totient function $\varphi(m)$ denotes the number of positive integers less than or equal to m which are coprime to m. [10 points]