The 24^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 4^{th} April 2014
Category II

Problem 1 *Let $f\colon (0,\infty) \to \mathbb{R}$ be a differentiable function. Assume that*

$$\lim_{x\to\infty} \left(f(x) + \frac{f'(x)}{x} \right) = 0\,.$$

Prove that

$$\lim_{x\to\infty} f(x) = 0\,.$$

Solution Assume that

$$\lim_{x\to\infty} \left(f(x) + \frac{f'(x)}{x} \right) = 0.$$

Fix $\varepsilon > 0$. Then there exists $x_0 > a$ such that

$$\left| f(x) + \frac{f'(x)}{x} \right| < \frac{\varepsilon}{2}$$

for all $x \geqslant x_0$. Take $x > x_0$ and define two functions $g, h\colon [x_0, x] \to \mathbb{R}$ by

$$g(x) = \mathrm{e}^{\frac{x^2}{2}} f(x) \;\text{ and }\; h(x) = \mathrm{e}^{\frac{x^2}{2}}.$$

Now by applying Cauchy's mean value theorem we get

$$\frac{g(x) - g(x_0)}{h(x) - h(x_0)} = \frac{g'(\eta)}{h'(\eta)} = f(\eta) + \frac{f'(\eta)}{\eta},$$

where $\eta \in (x_0, x)$. Hence

$$\left| f(x) - f(x_0)\mathrm{e}^{\frac{1}{2}(x_0^2 - x^2)} \right| = \left| f(\eta) + \frac{f'(\eta)}{\eta} \right| \cdot \left| 1 - \mathrm{e}^{\frac{1}{2}(x_0^2 - x^2)} \right|.$$

Consequently, for all $x > x_0$ one has

$$|f(x)| < \left| f(x_0)\mathrm{e}^{\frac{1}{2}(x_0^2 - x^2)} \right| + \frac{\varepsilon}{2} \left| 1 - \mathrm{e}^{\frac{1}{2}(x_0^2 - x^2)} \right| < |f(x_0)|\mathrm{e}^{\frac{1}{2}(x_0^2 - x^2)} + \frac{\varepsilon}{2}.$$

Since $\lim\limits_{x\to\infty} \mathrm{e}^{\frac{1}{2}(x_0^2 - x^2)} = 0$, there exists $\widetilde{x} \geqslant x_0$ such that

$$|f(x_0)|\mathrm{e}^{\frac{1}{2}(x_0^2 - x^2)} < \frac{\varepsilon}{2}$$

for all $x \geqslant \widetilde{x}$. Finally, taking into account the above considerations, we infer that

$$|f(x)| < \varepsilon$$

for all $x \geqslant \widetilde{x}$, which implies the desired conclusion. $\square$

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Problem 2 *Let p be a prime number and let A be a subgroup of the multiplicative group $\mathbb{F}_p^*$ of the finite field $\mathbb{F}_p$ with p elements. Prove that if the order of A is a multiple of 6, then there exist $x, y, z \in A$ satisfying $x + y = z$.*

Solution Obviously the existence of $x, y, z \in A$ with $x + y = z$ is equivalent to the existence of $x', y' \in A$ with $x' + y' = 1$ (just divide by z). First observe that A is cyclic, so $6 \mid d$ implies the existence of some $a \in A$ of order 6, hence $a^6 = 1$. This yields $a^3 = -1$, so that we obtain

$$a + a^3 + a^5 = a(1 + a^2 + a^4) = a\frac{1 - a^6}{1 - a^2} = 0,$$

from which we conclude

$$a + a^5 = -a^3 = 1.$$

Setting $x' := a$, $y' := a^5$ gives the desired equality. $\square$

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Problem 3 *Let k be a positive even integer. Show that*

$$\sum_{n=0}^{k/2}(-1)^n\binom{k+2}{n}\binom{2(k-n)+1}{k+1}=\frac{(k+1)(k+2)}{2}.$$

Solution

Lemma *For any positive integer k and $-1<x<1$ the following formula holds*

$$\left(\frac{1}{1-x}\right)^k=\sum_{n=0}^{\infty}\binom{k+n-1}{k-1}x^n.$$

Proof For $k=1$ we have

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=\sum_{n=0}^{\infty}\binom{n}{0}x^n.$$

Assume inductively that

$$\left(\frac{1}{1-x}\right)^k=\sum_{n=0}^{\infty}\binom{k+n-1}{k-1}x^n.$$

Then for $k+1$, by using the Cauchy product for two infinite series, we obtain

$$\left(\frac{1}{1-x}\right)^{k+1}=\left(\frac{1}{1-x}\right)^k\left(\frac{1}{1-x}\right)=\left(\sum_{n=0}^{\infty}\binom{k+n-1}{k-1}x^n\right)\left(\sum_{n=0}^{\infty}x^n\right)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{k+m-1}{k-1}x^n\right).$$

But, by using once again the induction principle, one can prove that

$$\sum_{m=0}^{n}\binom{k+m-1}{k-1}=\binom{k+n}{k}.$$

Finally, we obtain that

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{k+m-1}{k-1}x^n\right)=\sum_{n=0}^{\infty}\binom{k+n}{k}x^n,$$

which completes the proof of the lemma. □

Observe that

$$(1-x)^{k+2}=\frac{(1-x^2)^{k+2}}{(1+x)^{k+2}}=(1-x^2)^{k+2}\frac{1}{(1+x)^{k+2}},$$

for $-1<x<1$. On the other hand,

$$(1-x)^{k+2}=\sum_{n=0}^{k+2}(-1)^n\binom{k+2}{n}x^n,$$

$$(1-x^2)^{k+2}\frac{1}{(1+x)^{k+2}}=\left(\sum_{n=0}^{k+2}\binom{k+2}{n}(-1)^n x^{2n}\right)\left(\sum_{n=0}^{\infty}\binom{k+n+1}{k+1}(-1)^n x^n\right).$$

Consequently, by comparing the corresponding coefficients of x^k in the above equalities we obtain the following equation:

$$(-1)^k\binom{k+2}{k}=\sum_{n=0}^{k/2}\binom{k+2}{n}(-1)^n\binom{k+(k-2n)+1}{k+1}(-1)^{k-2n},$$

which implies the desired equality:

$$\sum_{n=0}^{k/2}(-1)^n\binom{k+2}{n}\binom{2(k-n)+1}{k+1}=\binom{k+2}{k}.$$

□

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Problem 4 *Let $0 < a < b$ and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function with $\int_a^b f(t)\,\mathrm{d}t = 0$. Show that*

$$\int_a^b \int_a^b f(x)f(y)\ln(x+y)\,\mathrm{d}x\,\mathrm{d}y \le 0\,.$$

Solution

Lemma *(well-known)*

$$\int_0^\infty \frac{e^{-t}-e^{-Kt}}{t}\,\mathrm{d}t = \log K$$

for every $K > 0$.

Proof Let

$$h(K) = \int_0^\infty \frac{e^{-t}-e^{-Kt}}{t}\,\mathrm{d}t.$$

From $h(1) = 0$ and $h'(K) = \displaystyle\int_0^\infty e^{-Kt}\,\mathrm{d}t = \frac{1}{K}$ we get

$$h(K) = \int_1^K \frac{\mathrm{d}k}{k} = \log K.$$

$\square$

To prove the problem statement, take the sum of the identities

$$\int_a^b \int_a^b f(x)f(y)\left(\int_0^\infty \frac{e^{-t}-e^{-(x+y)t}}{t}\,\mathrm{d}t\right)\mathrm{d}x\,\mathrm{d}y = \int_a^b \int_a^b f(x)f(y)\log(x+y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\int_a^b \int_a^b f(x)f(y)\left(\int_0^\infty \frac{-e^{-t}+e^{-(x+\frac{1}{2})t}}{t}\,\mathrm{d}t\right)\mathrm{d}x\,\mathrm{d}y = \int_a^b \int_0^\infty f(x)\frac{-e^{-t}+e^{-(x+\frac{1}{2})t}}{t}\,\mathrm{d}t\,\mathrm{d}x \cdot \int_a^b f(y)\,\mathrm{d}y = 0 \qquad \text{and}$$

$$\int_a^b \int_a^b f(x)f(y)\left(\int_0^\infty \frac{-e^{-t}+e^{-(y+\frac{1}{2})t}}{t}\,\mathrm{d}t\right)\mathrm{d}x\,\mathrm{d}y = \int_a^b \int_0^\infty f(y)\frac{-e^{-t}+e^{-(y+\frac{1}{2})t}}{t}\,\mathrm{d}t\,\mathrm{d}x \cdot \int_a^b f(x)\,\mathrm{d}x = 0.$$

As can be seen,

$$\begin{aligned}
\int_a^b \int_a^b f(x)f(y)\log(x+y)\,\mathrm{d}x\,\mathrm{d}y &= \int_a^b \int_a^b f(x)f(y)\left(\int_0^\infty \frac{-e^{-t}+e^{-(x+\frac{1}{2})t}+e^{-(y+\frac{1}{2})t}-e^{-(x+y)t}}{t}\,\mathrm{d}t\right)\mathrm{d}x\,\mathrm{d}y \\
&= -\int_0^\infty \int_a^b \int_a^b f(x)f(y)\frac{(e^{-\frac{1}{2}t}-e^{-xt})(e^{-\frac{1}{2}t}-e^{-yt})}{t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&= -\int_0^\infty \left(\int_a^b f(x)\big(e^{-\frac{1}{2}t}-e^{-xt}\big)\,\mathrm{d}x\right)^2 \frac{\mathrm{d}t}{t} \le 0.
\end{aligned}$$

$\square$