

The 24th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 4th April 2014
Category II

Problem 1 Let $f: (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function. Assume that

$$\lim_{x \rightarrow \infty} \left(f(x) + \frac{f'(x)}{x} \right) = 0.$$

Prove that

$$\lim_{x \rightarrow \infty} f(x) = 0.$$

Problem 2 Let p be a prime number and let A be a subgroup of the multiplicative group \mathbb{F}_p^* of the finite field \mathbb{F}_p with p elements. Prove that if the order of A is a multiple of 6, then there exist $x, y, z \in A$ satisfying $x + y = z$.

Problem 3 Let k be a positive even integer. Show that

$$\sum_{n=0}^{k/2} (-1)^n \binom{k+2}{n} \binom{2(k-n)+1}{k+1} = \frac{(k+1)(k+2)}{2}.$$

Problem 4 Let $0 < a < b$ and let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous function with $\int_a^b f(t) dt = 0$. Show that

$$\int_a^b \int_a^b f(x)f(y) \ln(x+y) dx dy \leq 0.$$