The 22nd Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 30th March 2012
Category II

Problem 1 *Let $f\colon [1,\infty) \to (0,\infty)$ be a non-increasing function such that*

$$\limsup_{n\to\infty} \frac{f(2^{n+1})}{f(2^n)} < \frac{1}{2}\,.$$

Prove that

$$\int_1^\infty f(x)\,\mathrm{d}x < \infty\,.$$

Problem 2 *Let M be the (tridiagonal) 10×10 matrix*

$$M = \begin{pmatrix} -1 & 3 & 0 & \cdots & \cdots & \cdots & 0 \\ 3 & 2 & -1 & 0 & & & \vdots \\ 0 & -1 & 2 & -1 & \ddots & & \vdots \\ \vdots & 0 & -1 & 2 & \ddots & 0 & \vdots \\ \vdots & & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & & & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Show that M has exactly nine positive real eigenvalues (counted with multiplicities).

Problem 3 *Let $(A,+,\cdot)$ be a ring with unity, having the following property: for all $x \in A$ either $x^2 = 1$ or $x^n = 0$ for some $n \in \mathbb{N}$. Show that A is a commutative ring.*

Problem 4 *Let a,b,c,x,y,z,t be positive real numbers with $1 \le x,y,z \le 4$. Prove that*

$$\frac{x}{(2a)^t} + \frac{y}{(2b)^t} + \frac{z}{(2c)^t} \ge \frac{y+z-x}{(b+c)^t} + \frac{z+x-y}{(c+a)^t} + \frac{x+y-z}{(a+b)^t}\,.$$