The 22$^{\text{nd}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 30$^{\text{th}}$ March 2012
Category I

**Problem 1** *Let* $f\colon [0,1] \to [0,1]$ *be a differentiable function such that* $|f'(x)| \neq 1$ *for all* $x \in [0,1]$. *Prove that there exist unique points* $\alpha, \beta \in [0,1]$ *such that* $f(\alpha) = \alpha$ *and* $f(\beta) = 1 - \beta$.

**Problem 2** *Determine all* $2 \times 2$ *integer matrices* $A$ *having the following properties:*

1. *the entries of* $A$ *are (positive) prime numbers,*
2. *there exists a* $2 \times 2$ *integer matrix* $B$ *such that* $A = B^2$ *and the determinant of* $B$ *is the square of a prime number.*

**Problem 3** *Determine the smallest real number* $C$ *such that the inequality*

$$\frac{x}{\sqrt{yz}} \cdot \frac{1}{x+1} + \frac{y}{\sqrt{zx}} \cdot \frac{1}{y+1} + \frac{z}{\sqrt{xy}} \cdot \frac{1}{z+1} \leq C$$

*holds for all positive real numbers* $x$, $y$ *and* $z$ *with*

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1\,.$$

**Problem 4** *Find all positive integers* $n$ *for which there exists a positive integer* $k$ *such that the decimal representation of* $n^k$ *starts and ends with the same digit.*