The 21st Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 31st March 2011
Category II

Problem 1 *Let $n > k$ and let $A_1, \dots, A_k$ be real $n \times n$ matrices of rank $n-1$. Prove that*

$$A_1 \cdot \ldots \cdot A_k \neq 0\,.$$

Solution Consider two linear operators $V \xrightarrow{g} V \xrightarrow{f} V$ of an n-dimensional vector space V. If $\mathrm{Ker}(f) \subset \mathrm{Im}(g)$, then $\dim(\mathrm{Im}\,(fg)) = \dim(\mathrm{Im}(g)) - \dim(\mathrm{Ker}(f))$. But we have the inequality

$$\dim(\mathrm{Im}(fg)) \geq \dim(\mathrm{Im}(g)) - \dim(\mathrm{Ker}(f))$$

in the general case. Applying the correspondence between linear operators and matrices, we obtain the inequality $\mathrm{rank}\,(AB) \geq \mathrm{rank}\,B - (n - \mathrm{rank}\,A)$ for every two matrices A and B. The inequality $\mathrm{rank}\,(A_1 \cdot \ldots \cdot A_k) \geq (\mathrm{rank}\,(A_1) + \ldots + \mathrm{rank}\,(A_k)) - (k-1)\,n$ can be deduced from the inequality $\mathrm{rank}\,(AB) \geq \mathrm{rank}\,A + \mathrm{rank}\,B - n$ by the simple induction. We obtain the inequality $\mathrm{rank}\,(A_1 \cdot \ldots \cdot A_k) \geq k\,(n-1) - (k-1)\,n = n - k$ in our case. Thus, if $k < n$ then $\mathrm{rank}\,(A_1 \cdot \ldots \cdot A_k) \geq 1$ and the product $A_1 \cdot \ldots \cdot A_k$ can not be equal to zero. □

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Problem 2 *Let k be a positive integer. Compute*

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{1}{n_1 n_2\dots n_k(n_1+\ldots+n_k+1)}.$$

Solution

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{1}{n_1 n_2\dots n_k(n_1+\ldots+n_k+1)}=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{1}{n_1 n_2\dots n_k}\int_0^1 x^{n_1+\ldots+n_k}\,\mathrm{dx}=$$

$$=\int_0^1\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{x^{n_1+\ldots+n_k}}{n_1 n_2\dots n_k}\,\mathrm{dx}=\int_0^1(-\log(1-x))^k\,\mathrm{dx}=[1-x=\mathrm{e}^{-u}]=\int_0^{\infty}u^k\mathrm{e}^{-u}\,\mathrm{du}=\Gamma(k+1)=k!$$

□

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Problem 3 *Let p and q be complex polynomials with $\deg p > \deg q$ and let $f(z) = \dfrac{p(z)}{q(z)}$. Suppose that all roots of p lie inside the unit circle $|z| = 1$ and that all roots of q lie outside the unit circle. Prove that*

$$\max_{|z|=1} |f'(z)| > \frac{\deg p - \deg q}{2} \max_{|z|=1} |f(z)|.$$

Solution Without loss of generality we can assume that the maximum of $|f|$ is attained at 1.

Let $p(z) = a \prod_{k=1}^{n_1} (z - c_k)$ and $q(z) = b \prod_{\ell=1}^{n_2} (z - d_\ell)$ where $n_1 = \deg p$ and $n_2 = \deg q$. Then

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n_1} \frac{1}{z - c_k} - \sum_{\ell=1}^{n_2} \frac{1}{z - d_\ell}\,.$$

Since $|c_k| < 1$ and $|d_\ell| > 1$ for all k and ℓ, we have

$$Re\, \frac{1}{1 - c_k} > \frac{1}{2}$$

and

$$Re\, \frac{1}{1 - d_k} < \frac{1}{2}.$$

Therefore,

$$\frac{|f'(1)|}{|f(1)|} \ge Re\, \frac{f'(1)}{f(1)} > n_1 \cdot \frac{1}{2} - n_2 \cdot \frac{1}{2} = \frac{\deg p - \deg q}{2}$$

and

$$\max_{|z|=1} |f'(z)| \ge |f'(1)| = \frac{|f'(1)|}{|f(1)|} \cdot |f(1)| \ge \frac{\deg p - \deg q}{2} \max_{|z|=1} |f(z)|\,.$$

□

Problem 4 *Let $\mathbb{Q}[x]$ denote the vector space over $\mathbb{Q}$ of polynomials with rational coefficients in one variable x. Find all $\mathbb{Q}$-linear maps $\Phi : \mathbb{Q}[x] \to \mathbb{Q}[x]$ such that for any irreducible polynomial $p \in \mathbb{Q}[x]$ the polynomial $\Phi(p)$ is also irreducible.*

(A polynomial $p \in \mathbb{Q}[x]$ is called irreducible if it is non-constant and the equality $p = q_1q_2$ is impossible for non-constant polynomials $q_1, q_2 \in \mathbb{Q}[x]$.)

Solution

The answer is $\Phi(p(x)) = ap(bx + c)$ for some non-zero rationals a, b and some rational c. It is clear that such operators preserve irreducibility. Let's prove that any irreducibility-preserving operator is of such form. We start with the following

Lemma 1 *Assume that $f, g \in \Pi$ are two polynomials such that for all rational numbers c the polynomial $f + cg$ is irreducible. Then either $g \equiv 0$, or f is non-constant linear polynomial and g is non-zero constant.*

Proof Let $g(x_0) \neq 0$ for some rational x_0. Then for $c = -f(x_0)/g(x_0)$ we have $(f + cg)(x_0) = 0$, so the polynomial $f + cg$ is divisible by $x - x_0$. Hence $f + cg = C(x - x_0)$ for some non-zero rational C. Choose $x_1 \neq x_0$ such that $g(x_1) \neq 0$. Then for $c_1 = -f(x_1)/g(x_1) \neq c$ (since $f(x_1) + cg(x_1) = C(x_1 - x_0) \neq 0$) we have $f + c_1g = C_1(x - x_1)$. Subtracting we get that $(c_1 - c)g$ is linear, hence g is linear, hence f too. If $f(x) = ax + b$, $g(x) = a_1x + b_1$, then $a \neq 0$ (since f is irreducible) and if $a_1 \neq 0$, then for $c = -a/a_1$ the polynomial $f + cg$ is constant, hence not irreducible. So $a_1 = 0$ and we are done. □

Now denote $g_k = \Phi(x^k)$.

Lemma 2 *g_0 is non-zero constant and g_1 is non-constant linear function.*

Proof Since $x+c$ is irreducible for any rational c, we get that g_1+cg_0 is irreducible for any rational c. By Lemma 1 we have that either $g_0 = 0$ or g_0 is constant and g_1 is linear non-constant. Assume that $g_0 = 0$. Note that for any rational α one may find rational β such that $x^2 + \alpha x + \beta$ is irreducible, hence $g_2 + \alpha g_1 = \Phi(x^2 + \alpha x + \beta)$ is irreducible for any rational α. It follows by Lemma 1 that g_1 is constant, hence not irreducible. A contradiction, hence $g_0 \neq 0$ and we are done. □

Denote $g_0 = C$, $g_1(x) = Ax + B$. Consider the new operator $p(x) \to C^{-1}\Phi(p(A^{-1}Cx - A^{-1}B))$. This operator of course preserves irreducibility, consider it instead Φ.

Now $g_0 = 1$, $g_1(x) = x$ and our goal is to prove that $g_n = x^n$ for all positive integers n. We use induction by n. Assume that $n \geq 2$ and $g_k(x) = x^k$ is already proved for $k = 0, 1, \ldots, n-1$. Denote $h(x) = g_n(x) - x^n$ and assume that h is not identical 0. For arbitrary monic irreducible polynomial f of degree n we have $\Phi(f) = f + h$, hence $f + h$ is irreducible aswell. Choose rational x_0 such that $h(x_0) \neq 0$, our goal is to find irreducible f such that $f(x_0) = -h(x_0)$ and hence $f + h$ has a root in x_0.

There are many ways to do it, consider one of them, via Eisenstein's criterion. Recall it.

Eisenstein's criterion *Assume that $f(x) = a_nx^n + \cdots + a_0$ is a polynomial with rational coefficients and p is a prime number so that $a_k = b_k/c_k$ with coprime integers b_k, c_k such that b_k is divisible by p for $k = 0, 1, \ldots, n-1$, both b_n and c_n are not divisible by p and b_0 is not divisible by p^2. Then f is irreducible.*

Without loss of generality, $x_0 = 0$ (else denote $x - x_0$ by new variable). Then we want to find an irreducible polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x - h(0)$. Denote $-h(0) = u/v$ for coprime positive integer v and non-zero integer u. Take $L = 6uv$ and consider the prime divisor p of the number $vL^n/u - 1$. Clearly, p does not divide $6uvL$. Then consider the polynomial $(x + L)^n - L^n + u/v$. If $vL^n/u - 1$ is not divisible by p^2, then we are done by Eisenstein's criterion (with new variable $y = x + L$). If $vL^n/u - 1$ is divisible by p^2, then add px to our polynomial and now Eisenstein's criterion works.

Unless $h(x) = -x^n + \ldots$, the polynomial $f + h$ is not linear and so is not irreducible. If $n \geq 3$, then we may add px^2 or $2px^2$ to our polynomial f and get non-linear $f + h$ (but still irreducible f). Finally, if $n = 2$, and $h(x) = -x^2 + ax + b$, then choose irreducible polynomial of the form $f(x) = x^2 - ax + c$ and get $f + h$ being constant (hence not irreducible).

The induction step and the whole proof are finished. □