The 21st Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 31st March 2011
Category II

Problem 1 *Let $n > k$ and let $A_1, \dots, A_k$ be real $n \times n$ matrices of rank $n-1$. Prove that*

$$A_1 \cdot \ldots \cdot A_k \neq 0\,.$$

Problem 2 *Let k be a positive integer. Compute*

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k (n_1 + \cdots + n_k + 1)}\,.$$

Problem 3 *Let p and q be complex polynomials with $\deg p > \deg q$ and let $f(z) = \dfrac{p(z)}{q(z)}$. Suppose that all roots of p lie inside the unit circle $|z| = 1$ and that all roots of q lie outside the unit circle. Prove that*

$$\max_{|z|=1} |f'(z)| > \frac{\deg p - \deg q}{2} \max_{|z|=1} |f(z)|\,.$$

Problem 4 *Let $\mathbb{Q}[x]$ denote the vector space over $\mathbb{Q}$ of polynomials with rational coefficients in one variable x. Find all $\mathbb{Q}$-linear maps $\Phi\colon \mathbb{Q}[x] \to \mathbb{Q}[x]$ such that for any irreducible polynomial $p \in \mathbb{Q}[x]$ the polynomial $\Phi(p)$ is also irreducible.*

(A polynomial $p \in \mathbb{Q}[x]$ is called irreducible if it is non-constant and the equality $p = q_1 q_2$ is impossible for non-constant polynomials $q_1, q_2 \in \mathbb{Q}[x]$.)