The 20$^{\text{th}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 25$^{\text{th}}$ March 2010
Category II

Problem 1 *Let a and b be given positive coprime integers. Then for every integer n there exist integers x, y such that*

$$n = ax + by\,.$$

Prove that $n = ab$ is the greatest integer for which $xy \leq 0$ in all such representations of n. [10 points]

Solution The greatest such integer is $a \cdot b$.

If $ab = ax + by$, then $a \mid y$ and $b \mid x$. Thus if $x > 0$, then $x \geq b$ and $by = ab - ax \leq ab - ab = 0$, so $y \leq 0$.

Now let $n > ab$. Let $n = ax + by$ be the representation such that x is positive and as small as possible. Then since $n = a(x - b) + b(y + a)$ is another representation of n, $x - b$ must not be positive and therefore $x \leq b$. Hence $by = n - ax \geq n - ab > 0$, so $y > 0$. $\square$

The 20th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 25th March 2010
Category II

Problem 2 *Prove or disprove that if a real sequence (a_n) satisfies $a_{n+1} - a_n \to 0$ and $a_{2n} - 2a_n \to 0$ as $n \to \infty$, then $a_n \to 0$.* [10 points]

Solution The proposition is true.

From the condition $a_{n+1} - a_n \to 0$ we conclude by Cesaro's lemma that $\frac{a_n}{n} \to 0$. Since the sequence $a_{2n} - 2a_n$ must be bounded, we know that

$$C := \sup\{|a_{2n} - 2a_n| : n \in \mathbb{N}\} < \infty .$$

Considering the identity

$$\frac{a_n}{n} - \frac{a_{n\cdot 2^{m+1}}}{n \cdot 2^{m+1}} = \sum_{k=0}^{m} \left(\frac{a_{n\cdot 2^k}}{n \cdot 2^k} - \frac{a_{n\cdot 2^{k+1}}}{n \cdot 2^{k+1}} \right)$$

we conclude by letting $m \to \infty$ and n fixed that

$$\frac{a_n}{n} = \sum_{k=0}^{\infty} \left(\frac{a_{n\cdot 2^k}}{n \cdot 2^k} - \frac{a_{n\cdot 2^{k+1}}}{n \cdot 2^{k+1}} \right) .$$

Now from

$$\left| \frac{a_n}{n} \right| \le \sum_{k=0}^{\infty} \left| \frac{a_{n\cdot 2^k}}{n \cdot 2^k} - \frac{a_{n\cdot 2^{k+1}}}{n \cdot 2^{k+1}} \right| \le \sum_{k=0}^{\infty} \frac{C}{n \cdot 2^{k+1}} = \frac{C}{n}$$

we infer that $|a_n| \le C$, i. e. the sequence (a_n) must be bounded.
Now suppose that (a_n) does not converge to 0. Then, by Bolzano's theorem, there must exist a subsequence (a_{n_k}) converging to some number $a \neq 0$. From the hypothesis we conclude in turn that

$$\begin{aligned} a_{2n_k} &\to 2a \,, \\ a_{4n_k} &\to 4a \,, \\ &\vdots \end{aligned}$$

which would result in an unbounded set of accumulation points $a, 2a, 4a, \ldots$ of (a_n) in contradiction to (a_n) being bounded. $\square$

Problem 3 *Let A and B be two $n \times n$ matrices with integer entries such that all of the matrices*

$$A\,, \quad A+B\,, \quad A+2B\,, \quad A+3B\,, \quad \dots\,, \quad A+(2n)B$$

are invertible and their inverses have integer entries, too. Show that $A+(2n+1)B$ is also invertible and that its inverse has integer entries. [10 points]

Solution Suppose that the $n \times n$ matrix M has integer entries and M has inverse matrix M^{-1} with integer entries. Then $M \cdot M^{-1} = I$ implies $\det M \cdot \det M^{-1} = 1$. Thus $\det M = 1$ or $\det M = -1$. Set $M(t) = A + tB$. The determinant of the matrix $M(t)$

$$\det M(t) = \det\,(A+tB) = \det A + \cdots + t^n \det B$$

is the polynomial of degree n in t. The polynomial $\det M(t)$ takes values 1 or -1 at points $t = 0, 1, 2, \dots, 2n$. Hence $\det M(t)$ takes the value 1 or the value -1 at least $n+1$ times. This implies that $\det M(t)$ is a constant polynomial: $M(t) = 1$ or $M(t) = -1$ for all t. Consequently, $\det M(2n+1) = \pm 1$. Hence the matrix $A+(2n+1)B$ is invertible. By Cramer's formula, the inverse matrix has integer entries, since the determinant is equal to 1 or -1. □

The 20$^{\text{th}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 25$^{\text{th}}$ March 2010
Category II

Problem 4 *Let $f\colon [0,1] \to \mathbb{R}$ be a function satisfying*

$$|f(x) - f(y)| \le |x - y|$$

for every $x, y \in [0,1]$. Show that for every $\varepsilon > 0$ there exists a countable family of rectangles (R_i) of dimensions $a_i \times b_i$, $a_i \le b_i$, in the plane such that

$$\big\{(x, f(x)) : x \in [0,1]\big\} \subset \bigcup_i R_i \quad \textit{and} \quad \sum_i a_i < \varepsilon\,.$$

(The edges of the rectangles are not necessarily parallel to the coordinate axes.) [10 points]

Solution Assume without loss of generality that $f(0) = 0$, thus $|f(x)| \le 1$ for $x \in [0,1]$.

First notice that if $C \subset [0,1]$ is a set of Lebesgue measure no larger than $\varepsilon/3$, then it can be covered by a countable family of intervals I_i of total measure at most $\varepsilon/2$, and thus $\{(x, f(x) : x \in C\}$ is covered by rectangles $I_i \times [-1,1]$, and their total width is at most $\varepsilon/2$.

Notice that as we are interested in only one dimension of the rectangle, and the graph we are to covered is bounded, we may as well think in terms of covering with strips instead of rectangles.

For now on fix $\varepsilon > 0$. We shall introduce a few definitions. Let $x, y \in [0,1]$. We say that the interval $[x,y]$ is covered, if $|f(z) - \alpha(z)| < \varepsilon|x-y|$ for all $z \in [x,y]$, where α is the linear function meeting f at x and y. The inclination of an interval $[x,y]$, denoted $i(x,y)$, is the number $|f(x) - f(y)|/|x-y|$. Notice the inclination of any interval cannot be larger than 1 as f is 1-Lipschitz.

Now we prove the following lemma.

Lemma *There exists a constant $\delta > 0$ such that the following holds. Consider any interval $[x,y] \subset [0,1]$. Then either $[x,y]$ is covered, or there exists a subinterval $[x',y'] \subset [x,y]$ of length $|y'-x'| > \delta|x-y|$ and inclination at least $i(x,y) + \varepsilon$.*

Proof The proof is pretty simple. If $[x,y]$ is not covered, then there exists a point $z \in [x,y]$ with $|f(z) - \alpha(z)| > \varepsilon|x-y|$. Without loss of generality assume $f(x) < f(y)$ and $f(z) - \alpha(z) > \varepsilon|x-y|$. The interval $[x,z]$ in this case has inclination

$$\begin{aligned} i(x,z) = |f(x) - f(z)|/|x-z| &= \frac{f(z)-f(x)}{z-x} \ge \frac{\alpha(z) + \varepsilon(y-x) - f(x)}{z-x} = \frac{\frac{f(y)-f(x)}{y-x}(z-x) + \varepsilon(y-x)}{z-x} \\ &= \frac{f(y)-f(x)}{x-y} + \varepsilon\frac{y-x}{z-x} \ge i(x,y) + \varepsilon. \end{aligned}$$

The cases of $f(x) > f(y)$ and (or) $f(z) - \alpha(z) < -\varepsilon|x-y|$ are similar. Moreover we have

$$f(z) > \alpha(z) + \varepsilon|x-y| = f(x) \pm i(x,y)(z-x) + \varepsilon(y-x)\,.$$

Thus

$$2|z-x| \ge |f(z) - f(x)| + i(x,y)|z-x| \ge f(z) - f(x) \pm i(x,y)(z-x) \ge \varepsilon|x-y|\,,$$

thus $|z-x| \ge \frac{\varepsilon|x-y|}{2}$, which finishes the proof of the lemma with $\delta = \varepsilon/2$. □

Take a constant $n > 1/\varepsilon$. If begin with an interval $[x,y]$ and apply the lemma n times, we end up with an interval of length at least $|x-y|\delta^n$, which is either covered, or has inclination at least $n\varepsilon$ — the second is impossible, however, as the inclination of any interval is at most 1. Thus for any interval we can find its subinterval of length at least δ^n times the length of the original, which is covered. Thus we have the following corollary: for any interval $[x,y] \subset [0,1]$ there exists a covered subinterval $[x',y']$ of $[x,y]$ of length at least $c|x-y|$ for some fixed constant c.

Now we are ready to solve the problem. We shall construct a family of disjoint intervals $C_i \subset [0,1]$, with the Lebesgue measure of $[0,1] \setminus \bigcup C_i$ no larger than ε. Each of these intervals will be covered, and thus we shall be able to cover the whole graph of f by rectangles — each interval is covered, and thus the appropriate piece of the graph is contained in a rectangle of width at most 2ε, while the remaining part can be covered by a countable family of vertical rectangles of total width at most 2ε. As ε was arbitrary, this will end the proof.

The construction of C_is follows directly from the corollary — we choose $C_0 = [x_0, y_0]$ to be the interval given by the corollary for $[0,1]$, then C_1 and C_2 the intervals for $[0, x_0]$ and $[y_0, 1]$ respectively, then (in the third step), C_3, C_4, C_5 and C_6 are given for $[0, x_1]$, $[y_1, x_0]$, $[y_0, x_2]$ and $[y_2, 1]$ respectively, and so on. In each step a constant fraction of measure is removed, thus after sufficiently many steps no more than ε measure remains. □