The 19th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 1st April 2009
Category II

Problem 1 *A positive integer m is called* self-descriptive in base b, *where $b \geq 2$ is an integer, if:*

i) The representation of m in base b is of the form $(a_0 a_1 \dots a_{b-1})_b$
(that is $m = a_0 b^{b-1} + a_1 b^{b-2} + \cdots + a_{b-2} b + a_{b-1}$, where $0 \leq a_i \leq b-1$ are integers).

ii) a_i is equal to the number of occurences of the number i in the sequence $(a_0 a_1 \dots a_{b-1})$.

For example, $(1210)_4$ is self-descriptive in base 4, because it has four digits and contains one 0, two 1s, one 2 and no 3s.

a) Find all bases $b \geq 2$ such that no number is self-descriptive in base b.

b) Prove that if x is a self-descriptive number in base b then the last (least significant) digit of x is 0.

[10 points]

Solution

1. For $b \geq 7$ it is easy to verify that the number of the form $(b-4)b^{b-1} + 2b^{b-2} + b^{b-3} + b^4$ is a self descriptive number (it contains $b-4$ instances of digit 0, two instances of digit 1, one instance of digit 2 and one instance of digit $b-4$), and numbers $21200_{(5)}$ and $2020_{(4)}$ are self-descriptive numbers in bases 5 and 4, respectively.

   It remains to show that for bases $2, 3$ and 6 no self descriptive numbers exist. First note, that a self-descriptive number (in any admissible base) contains at least one instance of the digit 0. If it does not, then the first digit is 0, which is a contradiction.

   It is easy to prove the claim for $b = 2, 3$.

   Let us prove it for $b = 6$. Assume there exists $x = (b_0 b_1 b_2 b_3 b_4 b_5)_{(6)}$, where x is a self-descriptive number. We observe the following about x:

   (a) $\sum_{i=0}^{5} b_i = 6$
   (b) $b_0 \neq 0$
   (c) $\sum_{i=1}^{5} b_i = |\{b_i, b_i \neq 0, i \geq 1\}| + 1$
   (d) Other than the first digit, the set of all other non-zero digits consists of several 1's and one 2.

   Observation 1d implies that all but one of the digits b_3, b_4 and b_5 are 0, now it is easy to check, that no such number is self-descriptive, which is a contradiction. Therefore base $b = 6$ contains no self-descriptive numbers.

2. Assume that there is in fact a self-descriptive number x in base b that it is b-digits long but not a multiple of b. The digit at position $b-1$ must be at least 1, meaning that there is at least one instance of the digit $b-1$ in x. At whatever position a that digit $b-1$ falls, there must be at least $b-1$ instances of digit a in x. Therefore, we have at least one instance of the digit 1, and $b-1$ instances of a. If $a > 1$, then x has more than b digits, leading to a contradiction of our initial statement. And if $a = 0$ or $a = 1$, that also leads to a contradiction.

3. These numbers are: 1210, 2020, 21200, 3211000, 42101000, 521001000, 6210001000. That these are the only such numbers, follows from previous observations.

□

Problem 2 *Let E be the set of all continuously differentiable real valued functions f on $[0,1]$ such that $f(0)=0$ and $f(1)=1$. Define*

$$J(f)=\int_0^1 (1+x^2)(f'(x))^2\,\mathrm{d}x\,.$$

a) Show that J achieves its minimum value at some element of E.

b) Calculate $\min_{f\in E} J(f)$.

[10 points]

Solution By the fundamental theorem of Calculus, we have

$$1=|f'(1)-f'(0)|=\Big|\int_0^1 f''(x)\,\mathrm{d}x\Big|\,.$$

Next, by using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\Big|\int_0^1 f''(x)\,\mathrm{d}x\Big| &= \Big|\int_0^1 \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} f''(x)\,\mathrm{d}x\Big| \\ &\le \Big(\int_0^1 (1+x^2)(f''(x))^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x\Big)^{1/2} \\ &= \Big(\int_0^1 (1+x^2)(f''(x))^2\,\mathrm{d}x\Big)^{1/2}\Big(\arctan x\Big|_0^1\Big)^{1/2} \\ &= \Big(\int_0^1 (1+x^2)(f''(x))^2\,\mathrm{d}x\Big)^{1/2}\frac{\sqrt{\pi}}{2}\,.\end{aligned}$$

Hence

$$\inf_{f\in E}\int_0^1 (1+x^2)(f''(x))^2\,\mathrm{d}x\ge\frac{4}{\pi}\,.$$

Finally, let

$$f(x):=\frac{4}{\pi}\int_0^x \arctan t\,\mathrm{d}t$$

for $x\in[0,1]$. Then $f'(x)=\frac{4}{\pi}\arctan x$ (by the fundamental theorem of Calculus) and $f''(x)=\frac{4}{\pi}\frac{1}{1+x^2}$, for $x\in[0,1]$. Consequently, we deduce that $f\in E$ and

$$J(f)=\int_0^1(1+x^2)\Big(\frac{4}{\pi}\frac{1}{1+x^2}\Big)^2\,\mathrm{d}x=\frac{16}{\pi^2}\int_0^1\frac{1}{1+x^2}\,\mathrm{d}x=\frac{16}{\pi^2}\cdot\frac{\pi}{4}=\frac{4}{\pi}\,,$$

which proves that J attains its minimum on E. This completes the solution. $\square$

Problem 3 *Let A be an $n \times n$ square matrix with integer entries. Suppose that $p^2A^{p^2} = q^2A^{q^2} + r^2I_n$ for some positive integers p, q, r where r is odd and $p^2 = q^2 + r^2$. Prove that $|\det A| = 1$.*
(Here I_n means the $n \times n$ identity matrix.) [10 points]

Solution Consider the function $f\colon \mathbb{R} \to \mathbb{R}$.

$$f(x) = p^2x^{p^2} - q^2x^{q^2} - r^2\,. \tag{1}$$

Observe that

$$f'(x) = p^4x^{q^2-1}\left(x^{r^2} - \left(\frac{q}{p}\right)^4\right).$$

The roots of equation $f'(x) = 0$ are $x_1 = 0$ and $x_2 = \left(\frac{q}{p}\right)^{\frac{4}{r^2}}$ ($r \neq 0$ and $q \neq 1$). From $f(0) = -r^2 < 0$ and $f\left(\left(\frac{q}{p}\right)^{\frac{4}{r^2}}\right) < 0$ we obtain

$$\operatorname{sgn} f(x) = \begin{cases} -1 & \text{if } x < 1\,, \\ 0 & \text{if } x = 1\,, \\ 1 & \text{if } x > 1\,. \end{cases} \tag{2}$$

So $x = 1$ is the only real root of equation $f(x) = 0$.

Since the matrix A verifies $f(A) = O_n$, some eigenvalue $\lambda \in \sigma_{\mathrm{P}}(A)$ satisfies the equation $f(\lambda) = 0$. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of the matrix A. We show that $|\lambda_k| \leq 1$ for all k. The fact $f(\lambda) = 0$ can be written as

$$p^2\lambda^{p^2} = q^2\lambda^{q^2} + r^2. \tag{3}$$

Passing the relation (3) at modulus we obtain $p^2|\lambda|^{p^2} \leq q^2|\lambda|^{q^2} + r^2$ or

$$f(|\lambda|) \leq 0\,. \tag{4}$$

From (2) and (4) we obtain $0 \leq |\lambda| \leq 1$ or $0 \leq |\lambda_k| \leq 1$ for all $k = 1, \dots, n$. Because $f(0) = -r^2 \neq 0$, it results that $\lambda_k \neq 0$ for all k.

Hence

$$0 < |\lambda_k| \leq 1 \quad \text{for all } k = 1, \dots, n\,. \tag{5}$$

From $\det A = \lambda_1\lambda_2\cdots\lambda_n$ we obtain

$$|\det A| = |\lambda_1\lambda_2\cdots\lambda_n| = |\lambda_1||\lambda_2|\cdots|\lambda_n| \leq 1\,. \tag{6}$$

From (5) and (6) we obtain

$$0 < |\det A| \leq 1\,. \tag{7}$$

Since $A \in M_n(\mathbb{Z})$, it follows that $|\det A| \in \mathbb{N}$. From (7) we obtain the conclusion that $|\det A| = 1$. $\square$

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Problem 4 *Let k, m, n be positive integers such that $1 \le m \le n$ and denote $S = \{1, 2, \ldots, n\}$. Suppose that $A_1, A_2, \ldots, A_k$ are m-element subsets of S with the following property: for every $i = 1, 2, \ldots, k$ there exists a partition $S = S_{1,i} \cup S_{2,i} \cup \cdots \cup S_{m,i}$ (into pairwise disjoint subsets) such that*

(i) A_i has precisely one element in common with each member of the above partition.

(ii) Every A_j, $j \neq i$ is disjoint from at least one member of the above partition.

Show that $k \le \binom{n-1}{m-1}$. [10 points]

Solution Without loss of generality assume that $1 \in S_1^{(i)}$ for all $i = 1, 2, \ldots, k$, because otherwise we simply rename members of each partition.

For every $i = 1, 2, \ldots, k$ define the polynomial

$$P_i(x_2, x_3, \ldots, x_n) = \prod_{l=2}^{m} \Big(\sum_{s \in S_l^{(i)}} x_s \Big)$$

and regard it as a polynomial over $\mathbb{R}$ in variables $x_2, x_3, \ldots, x_n$.

Observe that P_i is a homogenous polynomial of degree $m-1$ in $n-1$ variables. Also observe that all monomials in P_i are products of different x's, i.e. there are no monomials with squares or higher powers. The last statement follows simply from the fact that $S_2^{(i)}, \ldots, S_m^{(i)}$ are mutually disjoint. Such polynomials form a linear space over $\mathbb{R}$ of dimension $\binom{n-1}{m-1}$ and polynomials P_i belong to that space. If we prove that polynomials P_i, $i = 1, 2, \ldots, k$ are linearly independent, the inequality $k \le \binom{n-1}{m-1}$ will follow from the dimension argument.

For any $i = 1, 2, \ldots, k$ let χ_i be the characteristic vector of $A \cap \{2, 3, \ldots, n\}$. In other words, $\chi_i \in \{0, 1\}^{n-1}$ where the j-th coordinate of χ_i equals 1 if $j + 1 \in A$, and 0 otherwise.

For every i we know that each $A_i \cap S_l^{(i)}$ has exactly one element and therefore

$$P_i(\chi_i) = \prod_{l=2}^{m} |A_i \cap S_l^{(i)}| = \prod_{l=2}^{m} 1 = 1\,.$$

On the other hand, if $j \neq i$ then either some $A_j \cap S_l^{(i)}$, $l \ge 2$ is empty, or all $A_j \cap S_l^{(i)}$, $l \ge 2$ are nonempty but $A_j \cap S_1^{(i)} = \emptyset$. In the latter case we must have $|A_j \cap S_l^{(i)}| = 2$ for some $l \ge 2$. In any case we have at least one even factor in the following product, and so

$$P_i(\chi_j) = \prod_{l=2}^{m} |A_j \cap S_l^{(i)}| \equiv 0 \pmod 2\,.$$

Therefore all diagonal entries in the matrix $[P_i(\chi_j)]_{i,j=1,2,\ldots,k}$ are odd, while all non-diagonal entries are even. Consequently, its determinant is an odd integer, in particular it is not 0, and thus the matrix is regular. If polynomials P_i were linearly dependent, we would conclude that rows of $[P_i(\chi_j)]_{i,j=1,2,\ldots,k}$ are also linearly dependent, but this is not the case. Therefore P_i, $i = 1, 2, \ldots, k$ must be linearly independent and this completes the proof. □