The 19th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 1st April 2009
Category II

Problem 1 *A positive integer m is called* self-descriptive in base b, *where $b \geq 2$ is an integer, if:*

i) The representation of m in base b is of the form $(a_0 a_1 \dots a_{b-1})_b$ (that is $m = a_0 b^{b-1} + a_1 b^{b-2} + \dots + a_{b-2} b + a_{b-1}$, where $0 \leq a_i \leq b-1$ are integers).

ii) a_i is equal to the number of occurences of the number i in the sequence $(a_0 a_1 \dots a_{b-1})$.

For example, $(1210)_4$ is self-descriptive in base 4, because it has four digits and contains one 0, two 1s, one 2 and no 3s.

a) Find all bases $b \geq 2$ such that no number is self-descriptive in base b.

b) Prove that if x is a self-descriptive number in base b then the last (least significant) digit of x is 0.

[10 points]

Problem 2 *Let E be the set of all continuously differentiable real valued functions f on $[0,1]$ such that $f(0) = 0$ and $f(1) = 1$. Define*

$$J(f) = \int_0^1 (1 + x^2)(f'(x))^2 \, \mathrm{d}x \, .$$

a) Show that J achieves its minimum value at some element of E.

b) Calculate $\min\limits_{f \in E} J(f)$.

[10 points]

Problem 3 *Let A be an $n \times n$ square matrix with integer entries. Suppose that $p^2 A^{p^2} = q^2 A^{q^2} + r^2 I_n$ for some positive integers p, q, r where r is odd and $p^2 = q^2 + r^2$. Prove that $|\det A| = 1$. (Here I_n means the $n \times n$ identity matrix.)*

[10 points]

Problem 4 *Let k, m, n be positive integers such that $1 \leq m \leq n$ and denote $S = \{1, 2, \dots, n\}$. Suppose that $A_1, A_2, \dots, A_k$ are m-element subsets of S with the following property: for every $i = 1, 2, \dots, k$ there exists a partition $S = S_{1,i} \cup S_{2,i} \cup \dots \cup S_{m,i}$ (into pairwise disjoint subsets) such that*

(i) A_i has precisely one element in common with each member of the above partition.

(ii) Every A_j, $j \neq i$ is disjoint from at least one member of the above partition.

Show that $k \leq \binom{n-1}{m-1}$.

[10 points]