The 18th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 2nd April 2008
Category II

Problem 1. *Find all functions* $f\colon \mathbb{Z} \to \mathbb{Z}$ *such that*

$$19f(x) - 17f(f(x)) = 2x$$

for all $x \in \mathbb{Z}$. [10 points]

Problem 2. *Find all continuously differentiable functions* $f\colon [0,1] \to (0,\infty)$ *such that* $\frac{f(1)}{f(0)} = \mathrm{e}$ *and*

$$\int_0^1 \frac{\mathrm{d}x}{f(x)^2} + \int_0^1 f'(x)^2 \,\mathrm{d}x \le 2\,.$$

[10 points]

Problem 3. *Find all pairs of natural numbers* (n, m) *with* $1 < n < m$ *such that the numbers* 1, $\sqrt[n]{n}$ *and* $\sqrt[n]{m}$ *are* linearly dependent *over the field of rational numbers* $\mathbb{Q}$.
[10 points]

Problem 4. *We consider the following game for one person. The aim of the player is to reach a fixed capital* $C > 2$. *The player begins with capital* $0 < x_0 < C$. *In each turn let* x *be the player's current capital. Define* $s(x)$ *as follows:*

$$s(x) := \begin{cases} x & \text{if } x < 1 \\ C - x & \text{if } C - x < 1 \\ 1 & \text{otherwise.} \end{cases}$$

Then a fair coin is tossed and the player's capital either increases or decreases by $s(x)$, *each with probability* $\frac{1}{2}$. *Find the probability that in a finite number of turns the player wins by reaching the capital* C. [10 points]