Category II

Problem 1. *Construct a set* $A \subset [0,1] \times [0,1]$ *such that* A *is dense in* $[0,1] \times [0,1]$ *and every vertical and every horizontal line intersects* A *in at most one point.*

Solution. Take $\alpha, \beta \notin \mathbb{Q}$ such that $\frac{\alpha}{\beta} \notin \mathbb{Q}$. Then

$$A := \left\{ (\{n\alpha\}, \{n\beta\}) : n \in \mathbb{N} \right\},$$

where $\{x\}$ denotes the fractional part of x, fulfills the assumptions. □

Problem 2. *Let A be a real $n \times n$ matrix satisfying*

$$A + A^t = I\,,$$

where A^t denotes the transpose of A and I the $n \times n$ identity matrix. Show that $\det A > 0$.

Solution. The assumption $A + A^t = I$ is equivalent to saying $A = S + \frac{1}{2}I$ where S denotes an arbitrary real skew symmetric matrix. In particular, there exists some orthogonal matrix T that diagonalizes S and for which $D := T^tST$ contains the eigenvalues of S. They are either zero or purely imaginary and pairwise conjugated, i.e. of the form

$$r_1\mathrm{i}, -r_1\mathrm{i}, \ldots, r_s\mathrm{i}, -r_s\mathrm{i}, 0, \ldots, 0$$

with $r_k \in \mathbb{R}$ for all $k = 1, \ldots, s$. The determinant of A is evaluated as follows:

$$\det A = \det\Big(S + \frac{1}{2}I\Big) = \det\Big(D + \frac{1}{2}I\Big)$$

since $\det(T^tT) = 1$ and with the notations from above this expression is

$$\Big(\frac{1}{2}\Big)^{n-2s} \prod_{i=1}^{s} \Big(\frac{1}{2} + r_k\mathrm{i}\Big)\Big(\frac{1}{2} - r_k\mathrm{i}\Big) = \Big(\frac{1}{2}\Big)^{n-2s} \prod_{i=1}^{s} \Big(\frac{1}{4} + r_k^2\Big)\,.$$

As all factors are strictly positive the result follows. □

Problem 3. *Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1) = 0$. Prove that the set*

$$A := \{h \in [0,1] : f(x+h) = f(x) \text{ for some } x \in [0,1]\}$$

has Lebesgue measure at least $\frac{1}{2}$.

Solution. Let us observe, that if f is continuous then A is closed, thus A is Lebesgue measurable. Moreover the set

$$B := \{h \in [0,1] : 1 - h \in A\}$$

has the same Lebesgue measure as the set A. We show that $A \cup B = [0,1]$.

For any $h \in [0,1]$ we define a function $g\colon [0,1] \to \mathbb{R}$ by

$$g(x) = f(x+h) - f(x) \quad \text{if } x + h \leq 1$$

and

$$g(x) = f(x+h-1) - f(x) \quad \text{if } x + h > 1\,.$$

From the assumption we have that g is continuous. If f has its minimum and maximum, respectively, in x_0 and x_1, then $g(x_0) \geq 0$ and $g(x_1) \leq 0$. From Darboux property we have that, there exists x_2 such that $g(x_2) = 0$, therefore $h \in A$ or $h \in B$. This completes the proof. □

Problem 4. *Let S be a finite set with n elements and $\mathcal{F}$ a family of subsets of S with the following property:*

$$A \in \mathcal{F},\ A \subseteq B \subseteq S \Longrightarrow B \in \mathcal{F}$$

Prove that the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(t) := \sum_{A \in \mathcal{F}} t^{|A|}(1-t)^{|S\setminus A|}$$

is nondecreasing ($|A|$ denotes the number of elements of A).

Solution. Without loss of generality assume $S = \{1, 2, \ldots, n\}$. For each subset A and every $t \in [0,1]$ construct a set $I_{t,A} := \prod_{j=1}^{n} I_{t,A}^{(j)}$ in $\mathbb{R}^n$, where

$$I_{t,A}^{(j)} := \begin{cases} [0,t) & \text{if } j \in A \\ [t,1] & \text{if } j \notin A\,. \end{cases}$$

It's clear that for any two different subsets A and B the sets $I_{t,A}$ and $I_{t,B}$ are disjoint. Since the volume of $I_{t,A}$ is equal to $t^{|A|}(1-t)^{|A^c|}$ we have that $f(t)$ is equal to the volume of $\bigcup_{A\in\mathcal{F}} I_{t,A}$. So the claim will be proved if we prove that

$$\bigcup_{A\in\mathcal{F}} I_{t_1,A} \subseteq \bigcup_{A\in\mathcal{F}} I_{t_2,A} \quad \text{for all } 0 < t_1 < t_2 < 1\,. \tag{1}$$

Take an arbitrary $x = (x_1, x_2, \ldots, x_n) \in I_{t_1,A}$ for some $A \in \mathcal{F}$. Construct a set $B \subseteq S$ such that $j \in B$ if and only if $x_j \le t_2$. If $j \notin B$ then $x_j > t_2 > t_1$ which implies $j \notin A$. So $A \subseteq B$ and thus $B \in \mathcal{F}$. Moreover, from the definition of B, we have $x \in I_{t_2,B}$. This proves (1) and the problem is solved. □