The 17th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 28th March 2007
Category II

Problem 1. *Construct a set* $A \subset [0,1] \times [0,1]$ *such that* A *is dense in* $[0,1] \times [0,1]$ *and every vertical and every horizontal line intersects* A *in at most one point.* [10 points]

Problem 2. *Let* A *be a real* $n \times n$ *matrix satisfying*

$$A + A^t = I\,,$$

where A^t *denotes the transpose of* A *and* I *the* $n \times n$ *identity matrix. Show that* $\det A > 0$.
[10 points]

Problem 3. *Let* $f\colon [0,1] \to \mathbb{R}$ *be a continuous function such that* $f(0) = f(1) = 0$. *Prove that the set*

$$A := \big\{h \in [0,1] : f(x+h) = f(x) \text{ for some } x \in [0,1]\big\}$$

is Lebesgue measureable and has Lebesgue measure at least $\frac{1}{2}$. [10 points]

Problem 4. *Let* S *be a finite set with* n *elements and* $\mathcal{F}$ *a family of subsets of* S *with the following property:*

$$A \in \mathcal{F},\ A \subseteq B \subseteq S \Longrightarrow B \in \mathcal{F}.$$

Prove that the function $f\colon [0,1] \to \mathbb{R}$ *given by*

$$f(t) := \sum_{A \in \mathcal{F}} t^{|A|}(1-t)^{|S \setminus A|}$$

is nondecreasing ($|A|$ *denotes the number of elements of* A*).* [10 points]