The 16th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 29th March 2006
Category II

Problem 1.

(a) Let u and v be two nilpotent elements in a commutative ring (with or without unity). Prove that $u+v$ is also nilpotent. (An element u is called nilpotent if there exists a positive integer n for which $u^n = 0$.)

(b) Show an example of a (non-commutative) ring R and nilpotent elements $u, v \in R$ such that $u + v$ is not nilpotent.

[10 points]

Problem 2. Let $(G, \cdot)$ be a finite group of order n. Show that each element of G is a square if and only if n is odd. [10 points]

Problem 3. For a function $f\colon [0, 1] \to \mathbb{R}$ the secant of f at points $a, b \in [0, 1]$, $a < b$, is the line in $\mathbb{R}^2$ passing through $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$. A function is said to intersect its secant at a, b if there exists a point $c \in (a, b)$ such that $\big(c, f(c)\big)$ lies on the secant of f at a, b.

1. Find the set $\mathcal{F}$ of all continuous functions f such that for any $a, b \in [0, 1]$, $a < b$, the function f intersects its secant at a, b.

2. Does there exist a continuous function $f \notin \mathcal{F}$ such that for any rational $a, b \in [0, 1]$, $a < b$, the function f intersects its secant at a, b?

[10 points]

Problem 4. Let $f\colon [0, \infty) \to \mathbb{R}$ ba a strictly convex continuous function such that

$$\lim_{x\to+\infty} \frac{f(x)}{x} = +\infty\,.$$

Prove that the improper integral $\int_0^{+\infty} \sin\big(f(x)\big)\,\mathrm{d}x$ is convergent but not absolutely convergent. [10 points]