The 12th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 10th April 2002
Category II

Problem 1. Find all complex solutions to the system

$$\begin{aligned}(a+\mathrm{i}c)^3+(\mathrm{i}a+b)^3+(-b+\mathrm{i}c)^3&=-6\,,\\(a+\mathrm{i}c)^2+(\mathrm{i}a+b)^2+(-b+\mathrm{i}c)^2&=6\,,\\(1+\mathrm{i})a+2\mathrm{i}c&=0\,.\end{aligned}$$

[10 points]

Problem 2. A ring R (not necessarily commutative) contains at least one zero divisor and the number of zero divisors is finite. Prove that R is finite. [10 points]

Problem 3. Let E be the set of all continuous functions $u\colon[0,1]\to\mathbb{R}$ satisfying

$$u^2(t)\le 1+4\int_0^t s\big|u(s)\big|\,\mathrm{d}s\,,\qquad \forall t\in[0,1]\,.$$

Let $\varphi\colon E\to\mathbb{R}$ be defined by

$$\varphi(u)=\int_0^1\big(u^2(x)-u(x)\big)\,\mathrm{d}x\,.$$

Prove that φ has a maximum value and find it. [10 points]

Problem 4. Prove that

$$\lim_{n\to\infty} n^2\left(\int_0^1\sqrt[n]{1+x^n}\,\mathrm{d}x-1\right)=\frac{\pi^2}{12}\,.$$

[10 points]