The 12th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 10th April 2002
Category I

Problem 1. Differentiable functions $f_1, \dots, f_n \colon \mathbb{R} \to \mathbb{R}$ are linearly independent. Prove that there exist at least $n-1$ linearly independent functions among $f_1', \dots, f_n'$. [10 points]

Problem 2. Let $p > 3$ be a prime number and $n = (2^{2p} - 1)/3$. Show that n divides $2^n - 2$. [10 points]

Problem 3. Positive numbers $x_1, \dots, x_n$ satisfy

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \dots + \frac{1}{1+x_n} = 1\,.$$

Prove that

$$\sqrt{x_1} + \sqrt{x_2} + \dots + \sqrt{x_n} \geq (n-1)\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \dots + \frac{1}{\sqrt{x_n}}\right).$$

[10 points]

Problem 4. The numbers $1, 2, \dots, n$ are assigned to the vertices of a regular n-gon in an arbitrary order. For each edge compute the product of the two numbers at the endpoints and sum up these products. What is the smallest possible value of this sum?
[10 points]