The 11$^{\text{th}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 4$^{\text{th}}$ April 2001
Category I

Problem 1 *Prove that for an arbitrary prime* $p \geq 5$ *the number*

$$\sum_{0<k<\frac{2p}{3}} \binom{p}{k}$$

is divisible by p^2.

Solution The number $\frac{1}{p}\binom{p}{k}$ for $0 < k < \frac{2p}{3}$ is congruent to $(-1)^{k-1}\frac{1}{k}$ modulo p. Hence it is sufficient to show that the element

$$\sum(1) = \sum_{0<k<2p/3} (-1)^{k-1}\frac{1}{k}$$

is 0 in a finite field F_p. The sum

$$\sum(2) = \sum_{k<\frac{p}{6}}\left(\frac{1}{k} - \frac{1}{2k} - \frac{1}{2k}\right) + \sum_{\frac{p}{6}<k<\frac{p}{3}}\left(\frac{1}{k} - \frac{1}{2k} + \frac{1}{p-2k}\right)$$

is 0 in F_p. But $\sum(1) = \sum(2)$. In fact the terms of the shape $\frac{1}{2k+1}$ are evidently the same. As to a term $\frac{1}{2k}$ in the $\sum(2)$ for $2k < \frac{p}{3}$ it appears with the coefficient -1, what is O.K. The term $\frac{1}{2k}$ (for $2k < \frac{p}{3}$) appears in $\sum(2)$ twice with the sign "$-$" and once with the sign "$+$". So $\sum(1) = \sum(2)$. □

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Category I

Problem 2 *Let $n \geq 2$ be a natural number. Prove that*

$$\prod_{k=2}^{n} \ln k < \frac{\sqrt{n!}}{n} .$$

Solution Consider $f\colon [1,\infty) \to \mathbb{R}$ defined by

$$f(t) = 2\ln t - t + \frac{1}{t} .$$

We have

$$f'(t) = -\frac{(t-1)^2}{t^2} ,$$

hence f' is negative on $(1,\infty)$. Therefore f is strictly decreasing. Since $f(1) = 0$, the function f has negative values on $(1,\infty)$. So

$$(\forall t \in (1,\infty)) \quad \Big(2\ln t - t + \frac{1}{t}\Big)\frac{1}{t^2-1} < 0 .$$

Putting $x = t^2$ in the above inequality we obtain that

$$(\forall t \in (1,\infty)) \quad \frac{\ln x}{x-1} < \frac{1}{\sqrt{x}} .$$

Hence

$$\prod_{k=2}^{n} \frac{\ln k}{k-1} < \prod_{k=2}^{n} \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{n!}}$$

and

$$\prod_{k=2}^{n} \ln k < \frac{(n-1)!}{\sqrt{n!}} = \frac{\sqrt{n!}}{n} .$$

□

Solution We prove that $\ln k < \sqrt{k} \cdot \frac{k-1}{k}$ for $k \in \mathbb{N}$ and ≥ 2. Let us consider the functions $f(x) = \ln x$ and $g(x) = \sqrt{x} \cdot \frac{x-1}{x}$. It is

$$f(1) = g(1) = 0 \quad \text{and} \quad f'(x) = \frac{1}{x} ,$$
$$g'(x) = \frac{1}{2\sqrt{x}}\frac{x-1}{x} + \sqrt{x}\frac{1}{x^2} ,$$
$$g'(x) - f'(x) = \frac{1}{2\sqrt{x^3}}\big(x - 1 + 2 - 2\sqrt{x}\big) = \frac{1}{2\sqrt{x^3}}\big(\sqrt{x}-1\big)^2,$$

which is $= 0$ for $x = 1$ and > 0 for $x > 1$. It proves that $g(x) > f(x)$ for $x > 1$, as we needed. Now by multiplying $\ln k < \sqrt{k} \cdot \frac{k-1}{k}$ over $k = 2, 3, \ldots, n$ we get

$$\prod_{k=2}^{n} \ln k < \left(\prod_{k=2}^{n} \sqrt{k}\right) \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \ldots \cdot \frac{n-1}{n} = \sqrt{n!} \cdot \frac{1}{n} ,$$

the problem solved.

□

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Category I

Problem 3 *Let A, B, C be sets in $\mathbb{R}^n$. Suppose that A is nonempty and bounded, that C is closed and convex, and that $A + B \subseteq A + C$. Show the inclusion $B \subseteq C$.*

We remind you that

$$E + F = \{e + f : e \in E, f \in F\}$$

and $D \subseteq \mathbb{R}^n$ is convex when

$$\forall x, y \in D\, \forall t \in [0, 1] : tx + (1 - t)y \in D\,.$$

Solution We will use the following lemma.

Lemma *Let $a_1, \ldots, a_m$ be points of a convex set $D \subseteq \mathbb{R}^n$. Let $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$. Then $\lambda_1 a_1 + \cdots + \lambda_m a_m \in D$.*

Proof We argue by induction on m. When $m = 1$ the assertion is trivial. Suppose that the assertion holds when m is some positive integer k. Let

$$x = \lambda_1 a_1 + \cdots + \lambda_{k+1} a_{k+1}\,,$$

where $a_1, \ldots, a_{k+1} \in D$ and $\lambda_1, \ldots, \lambda_{k+1} \geq 0$ with $\lambda_1 + \cdots + \lambda_{k+1} = 1$. At least one λ_i must be less than 1, say $\lambda_{k+1} < 1$. Write

$$y = \frac{\lambda_1}{\lambda} a_1 + \cdots + \frac{\lambda_k}{\lambda} a_k\,,$$

where

$$\lambda = \lambda_1 + \cdots + \lambda_k = 1 - \lambda_{k+1} > 0\,.$$

By the induction hypothesis, $y \in D$. Since D is convex and contains both y and a_{k+1} the equation $x = \lambda y + \lambda_{k+1} a_{k+1}$ shows that $x \in D$. This completes the proof by induction. □

Let $a_0 \in A$. If $b \in B$, then $a_0 + b \in A + B \subseteq A + C$, and so there exists $a_1 \in A, c_1 \in C$ such that $a_0 + b = a_1 + c_1$. Similarly, there exist $a_2, \ldots, a_i \in A$ and $c_2, \ldots, c_i \in C$ with

$$a_1 + b = a_2 + c_2, \ldots, a_{i-1} + b = a_i + c_i\,.$$

We add the i equations above together to deduce that

$$a_0 + ib = a_1 + c_1 + \cdots + c_i\,.$$

Since C is convex, the point x_i defined by the equation

$$x_i = \frac{1}{i}(c_1 + \cdots + c_i)$$

lies in C (Lemma). Now

$$\|b - x_i\| = \frac{1}{i}\|a_i - a_0\| \to 0 \quad \text{as} \quad i \to \infty\,,$$

since A is bounded. Thus $x_i \to b$ as $i \to \infty$. But C is closed, whence $b \in C$ and $B \subseteq C$. □

Solution For contradiction suppose there is $b \in B$ which $\notin C$. Since C is convex and closed, there existS $(n-1)$-dimensional hyperplane H such that it separates b and C. Denote $\vec{n}$ the normal vector of H orientated in direction of point b. Now every point x of space $\mathbb{R}^n$ can be expressed as $x = h_x + a\vec{n}$, where $h_x \in H$ and $a \in \mathbb{R}$. From this define linear function $f(x) = a$. It is clear that $f(b) > 0$ and $f(C) < 0$. Take now $\sup_{a \in A} f(a)$ (it is finite since A is bounded) and point a_0 such that $f(a_0) > \sup_{a \in A} f(a) - f(b)$. Then clearly, since function f is linear, it holds

$$f(a_0 + b) = f(a_0) + f(b) > \Big(f(a) - f(b)\Big) + f(b) > f(a) + f(c) = f(a + c)\,,$$

for all $a \in A$ and $c \in C$. But it is contradiction with $f(A+B) \subseteq f(A+C)$ (which follows from $A+B \subseteq A+C$). □

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Category I

Problem 4 *Let A be a set of positive integers greater than 0 such that for any $x, y \in A$, $x > y$,*

$$x - y \geq \frac{xy}{25}.$$

Find the maximal possible number of elements of the set A.

Solution For $x > y \geq 25$ we have

$$x - y < x \leq \frac{xy}{25}.$$

Hence A contains at most one element greater than 24. Let $A = \{x_1, x_2, \ldots, x_n\}$ where $x_1 < x_2 < \cdots < x_n$, $x_{n-1} < 25$. For the differences $d_j = x_{j+1} - x_j$, $1 \leq j \leq n-1$, we get

$$d_j \geq \frac{x_{j+1}x_j}{25} = \frac{(x_j + d_j)x_j}{25},$$

which yields

$$d_j \geq \frac{x_j^2}{25 - x_j}.$$

Since the function $g(x) = \frac{x^2}{25-x}$ is increasing in the interval $[0, 25)$, we obtain succesively

$$\begin{array}{ll} x_5 \geq 5, & d_5 \geq g(5) > 1, \\ x_6 \geq 7, & d_6 \geq g(7) > 2, \\ x_7 \geq 10, & d_7 \geq g(10) > 6, \\ x_8 \geq 17, & d_8 \geq g(17) > 36, \\ x_9 \geq 54. & \end{array}$$

So, we get $n \leq 9$. Simultaneously, we can see that the set with 9 elements

$$A = \{1, 2, 3, 4, 5, 7, 10, 17, 54\}$$

satisfies all the conditions. □

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Category II

Problem 1 *Let $n \geq 2$ be an integer and let $x_1, x_2, \ldots, x_n$ be real numbers. Consider $N = \binom{n}{2}$ sums $x_i + x_j$, $1 \leq i < j \leq n$ and denote them by $y_1, y_2, \ldots, y_N$ (in arbitrary order). For which n are the numbers $x_1, x_2, \ldots, x_n$ uniquely determined by the numbers $y_1, y_2, \ldots, y_N$?*

Solution The answer is $n \neq 2^p$.

Denote the kth symmetric polynomial in $x_1, x_2, \ldots, x_n$ by σ_k. Further denote

$$s_k = \sum_{i=1}^{n} x_i^k, \quad t_k = \sum_{i=1}^{N} y_i^k \,.$$

The numbers $x_1, x_2, \ldots, x_n$ are uniquely determined by the numbers $\sigma_1, \sigma_2, \ldots, \sigma_n$ and these are uniquely determined by the numbers $s_1, s_2, \ldots, s_n$ since we have the following identity:

$$s_k - s_{k-1}\sigma_1 + s_{k-2}\sigma_2 - \cdots + (-1)^{k-1} s_1 \sigma_{k-1} = (-1)^{k-1} k \sigma_k \,.$$

So we will try to show that $s_1, s_2, \ldots, s_n$ are determined by the numbers $t_1, t_2, \ldots, t_n$. We have

$$2t_k + 2^k s_k = \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i + x_j)^k = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=0}^{k}\binom{k}{r} x_i^r x_j^{k-r} =$$

$$= 2n s_k + \sum_{r=1}^{k-1}\binom{k}{r} s_r s_{k-r} \,.$$

For $n \neq 2^{k-1}$, we get

$$s_k = \frac{1}{2n - 2^k}\Big(2t_k - \sum_{r=1}^{k-1}\binom{k}{r} s_r s_{k-r}\Big) \,.$$

Using induction with respect to k, we can conclude that for $n \neq 2^p$, the numbers $t_1, t_2, \ldots, t_n$ determine uniquely the numbers $s_1, s_2, \ldots, s_n$.

For $n = 2$ the numbers from the sets $A_2 = \{0, 3\}$ and $B_2 = \{1, 2\}$ have the same sums. Suppose that we have two disjoint sets A_n, B_n, every with n elements, which have the same sums of all possible couples. Then the sets $A_{2n} = A_n \cup (c + B_n)$ and $B_{2n} = B_n \cup (c + A_n)$ for c large enought are disjoint with $2n$ elements and have the same sums of all possible couples. $\square$

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Category II

Problem 2 *Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function of function $\{f_n\}$, $f_n\colon [0,1] \to \mathbb{R}$. Define the sequence in the following way:*

$$f_{n+1}(x) = \int_0^x f_t,\ n = 0, 1, 2, \dots .$$

Prove that if $f_n(1) = 0$ for all n, then $f(x) \equiv 0$.

Solution Using induction on k, we prove that for any $n, k \geq 0$ integers

$$\int_0^1 (1-t)^k f_n(t) = k! \cdot f_{n+k}(1)\,. \tag{1}$$

This is trivial for $k = 0$. For greater k,

$$\int_0^1 (1-t)^k f_n(t) = [(1-t)^k f_{n+1}(t)]_{t=0}^1 + k \int_0^1 (1-t)^{k-1} f_{n+1}(t) = \\ = 0 + k \cdot (k-1)! \cdot f_{(n+1)+(k-1)}(1) = k! \cdot f_{n+k}(1)\,.$$

From (1) it follows for an arbitrary polynomial p, that $\int_0^1 p \cdot f = 0$.

By Weierstrass' theorem, for an arbitrary $\varepsilon > 0$ there exists a polynomial p_ε such that $|p_\varepsilon(t) - f(t)| < \varepsilon$ for all $t \in [0,1]$. This implies

$$\int_0^1 f^2 = \int_0^1 f^2 - \int_0^1 p_\varepsilon \cdot f = \int_0^1 (f - p_\varepsilon) f \leq \varepsilon \int_0^1 |f|\,.$$

This holds for any ε, thus $\int_0^1 f^2 = 0$. This implies $f \equiv 0$. □

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Problem 3 *Let $f\colon (0,+\infty) \to (0,+\infty)$ be a decreasing function, satisfying*

$$\int_0^\infty f(x) < \infty\,.$$

Prove that $\lim\limits_{x\to\infty} xf(x) = 0$.

Solution As first we prove that $\liminf\limits_{x\to\infty} xf(x) = 0$. Let $\liminf\limits_{x\to\infty} xf(x) = c > 0$, that implies $\exists x_0 \forall x > x_0 : xf(x) > c' > 0$, or $f(x) > \frac{c'}{x}$, and we get:

$$\int_0^\infty f(x) > \int_{x_0}^\infty f(x) > \int_{x_0}^\infty \frac{c'}{x} = \infty,$$

a contradiction.

Now, let us suppose $\limsup\limits_{x\to\infty} xf(x) = c > 0$. It implies $\forall y \exists x > y : xf(x) \geq \frac{c}{2}$. We have also constructed a sequence $\{x_n\}_{n=1}^\infty$, satisfying:

$$x_n \to \infty, \text{ and } x_n f(x_n) \geq \frac{c}{2} > 0, \text{ which is the same as } f(x_n) \geq \frac{c}{2x_n}\,.$$

Since f is decreasing: $f(x) > f(x_n)$, for $x \in (x_{n-1}, x_n]$ and

$$\infty > \int_0^\infty f(x) > \sum_{n=1}^\infty (x_n - x_{n-1}) f(x_n) \geq$$
$$\geq \frac{c}{2} \sum_{n=1}^\infty \frac{x_n - x_{n-1}}{x_n} = \frac{c}{2} \sum_{n=D1}^\infty \left(1 - \frac{x_{n-1}}{x_n}\right).$$

So we have a sequence $\{x_n\}_{n=1}^\infty$ such that $x_n \to \infty$ and $\sum\limits_{n=1}^\infty \left(1 - \frac{x_{n-1}}{x_n}\right) < \infty$.

To make a proof clearer, we will do a substitution $b_n = 1 - \frac{x_{n-1}}{x_n}$. Sequence $\{b_n\}_{n=1}^\infty$ satisfies:

$$\sum_{n=1}^\infty b_n < \infty \quad \text{and} \quad \prod_{n=1}^\infty (1 - b_n) = \lim_{n\to\infty} \frac{x_0}{x_n} = 0\,.$$

Second condition for a sequence $\{b_n\}_{n=1}^\infty$ is the same as $\sum\limits_{n=1}^\infty -\ln(1 - b_n) = \infty$.

From the ratio criterion for convergence of the infinity sums, if

$$\sum_{n=1}^\infty b_n < \infty \quad \text{and} \quad \sum_{n=1}^\infty -\ln(1 - b_n) = \infty,$$
$$\lim_{n\to\infty} \frac{-\ln(1 - b_n)}{b_n} = \infty$$

should hold. But above limit is equal to 1, as can be easy checked by many ways. (From $\sum\limits_{n=1}^\infty b_n < \infty$, we get $b_n \to 0$, and

$$\lim_{n\to\infty} \frac{-\ln(1-b_n)}{b_n} = \lim_{b_n\to 0} \frac{-\ln(1-b_n)}{b_n} \overset{\text{L'H}}{=} \lim_{b_n\to 0} \frac{\frac{1}{1-b_n}}{1} = 1\,.)$$

This yields to contradiction.

As a conclusion we have $\liminf\limits_{x\to\infty} xf(x) = 0$ and $\limsup\limits_{x\to\infty} xf(x) = 0$. □

Solution For contradiction assume that $\lim\limits_{x\to\infty} xf(x) = 0$ is not true. Then it must exist increasing sequence $\{x_i\}_{i=1}^\infty$, $x_i \to \infty$, such that exists $\varepsilon > 0$ that $x_i f(x_i) > \varepsilon$ for all x_i. Moreover, we can choose subsequence

$\{y_i\} \subset \{x_i\}$, such that $y_{i+1} \geq 2y_i$. Then following inequalities hold (inequality $(*)$ holds, because f is decreasing function):

$$\int_0^\infty f(x) \overset{(*)}{\geq} \sum_{n=2}^{\infty} (y_n - y_{n-1}) f(y_n) \geq \frac{1}{2} \sum_{n=2}^{\infty} y_n f(y_n) \geq \frac{1}{2} \sum_{n=2}^{\infty} \varepsilon = \infty\,.$$

This contradicts the assumption that $\int\limits_0^\infty f(x) < \infty$. □

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Problem 4 *Let R be an associative non commutative ring and let $n > 2$ be a fixed natural number. Assume that $x^n = x$, $\forall x \in R$. Prove that $xy^{n-1} = y^{n-1}x$ holds $\forall x, y \in R$.*

Solution Let $a = x^{n-1}$, then

$$a^2 = (x^{n-1})^2 = x^{2n-2} = x^n x^{n-2} = xx^{n-2} = x^{n-1} = a.$$

We show that if $r^2 = 0$ then $r = 0$. Indeed $r = r^n = r^{n-2}r^2 = 0$. If $e^2 = e$ then for every $x \in R$:

$$(ex - exe)^2 = (ex - exe)(ex - exe) =$$
$$= exex - exexe - exe^2x + exe^2xe = exex - exexe - exex + exexe = 0$$

and similarly

$$(ex - exe)^2 = 0$$

so $ex - exe = 0$ and $xe - exe = 0$, so for every $x \in R$ and every $e \in R$, such that $e^2 = e$ we have:

$$ex = xe$$

and since for every $y \in R$, $(y^{n-1}) = y^{n-1}$, we get:

$$xy^{n-1} = y^{n-1}x$$

for every $x, y \in R$. □

Solution Since R is an integral domain and

$$y(xy^{n-1} - y^{n-1}x)y = yxy^n - y^n xy = yxy - yxy = 0\,,$$

it is either $xy^{n-1} - y^{n-1}x = 0$ or $y = 0$, but that also implies $xy^{n-1} = y^{n-1}x$. The end. □