The 11th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 4th April 2001
Category I

Problem 1. Let A be a set of positive integers such that for any $x, y \in A$

$$x > y \implies x - y \geq \frac{xy}{25}.$$

Find the maximal possible number of elements of the set A.

[10 points]

Problem 2. Prove that for any prime $p \geq 5$, the number

$$\sum_{0<k<\frac{2p}{3}} \binom{p}{k}$$

is divisible by p^2. [10 points]

Problem 3. Let $n \geq 2$ be a natural number. Prove that

$$\prod_{k=2}^{n} \ln k < \frac{\sqrt{n!}}{n}.$$

[10 points]

Problem 4. Let A, B, C be nonempty sets in $\mathbb{R}^n$. Suppose that A is bounded, C is closed and convex, and $A + B \subseteq A + C$. Prove that $B \subseteq C$.

We remind that $E + F = \{ e + f : e \in E, f \in F \}$ and $D \subseteq \mathbb{R}^n$ is convex iff $tx + (1-t)y \in D$ for all $x, y \in D$ and any $t \in [0, 1]$.

[10 points]