The 10th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 5th April 2000
Category II

Problem 1. Let p be a prime of the form $p = 4n - 1$ where n is a positive integer. Prove that

$$\prod_{k=1}^{p} (k^2 + 1) \equiv 4 \pmod{p}.$$

[10 points]

Problem 2. If we write the sequence AAABABBB along the perimeter of a circle, then every word of the length 3 consisting of letters A and B (i.e. AAA, AAB, ABA, BAB, ABB, BBB, BBA, BAA) occurs exactly once on the perimeter. Decide whether it is possible to write a sequence of letters from a k-element alphabet along the perimeter of a circle in such a way that every word of the length l (i.e. an ordered l-tuple of letters) occurs exactly once on the perimeter. [10 points]

Problem 3. Let m, n be positive integers and let $x \in [0, 1]$. Prove that

$$(1 - x^n)^m + \big(1 - (1 - x)^m\big)^n \geq 1.$$

[10 points]

Problem 4. Let $\mathcal{B}$ be a family of open balls in $\mathbb{R}^n$ and $c < \lambda\big(\bigcup \mathcal{B}\big)$ where λ is the n-dimensional Lebesgue measure. Show that there exists a finite family of pairwise disjoint balls $\{U_i\}_{i=1}^k \subseteq \mathcal{B}$ such that

$$\sum_{j=1}^{k} \lambda(U_j) > \frac{c}{3^n}.$$

[10 points]