The 8^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 1^{st} April 1998
Category I

Problem 1 *Let a and d be two positive integers. Prove that there exists a constant K such that every set of K consecutive elements of the arithmetic progression $\{a+nd\}_{n=1}^{\infty}$ contains at least one number which is not prime.*

Solution Let p be a prime number such that $p \nmid d$. Let for some n all elements $a_n, \ldots, a_{n+p}$ are prime numbers. Clearly for $i = 1, \ldots, p$ we have $a_{n+i} = a_n + id$. Thus $a_n > p$, because $a_{n+a_n} = a_n + a_n d$ it is not a prime number. Consider now the system of the reminders $a_n \mod p, \ldots, a_{n+p} \pmod p$. Because all are primes greater than p, this system does not contain 0. Thus there exist two numbers $r_1 < r_2$ such that $a_{r_1} \equiv a_{r_2} \pmod p$. This yields $a_n + r_2 d \equiv a_n + r_1 d \pmod p$, and so $r_2 d \equiv r_1 d \pmod p$. But $(p, d) = 1$ therefore $r_2 \equiv r_1 \pmod p$ which is a contradiction. □

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Problem 2 *Find the limit*

$$\lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)^n}{\mathrm{e}}\right)^n.$$

Solution Recall that

$$\lim_{n\to\infty}\left(1\pm\frac{1}{n}\right)^n=\mathrm{e}^{\pm 1},$$

and put

$$\begin{aligned} f(x)&=(1+x)^{\frac{1}{x}}\quad \text{for} x>0\,,\\ f(0)&=\lim_{x\to 0^+}f(x)=e\,. \end{aligned}$$

It is easy to verify that $f'_+(0)=-\frac{\mathrm{e}}{2}$. Hence

$$f(x)=\mathrm{e}(1-2x)+o(x)\quad\text{and}\quad \frac{(1+\frac{1}{n})^n}{\mathrm{e}}=1-\frac{1}{2n}+o\Big(\frac{1}{n}\Big)$$

and

$$\lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)^n}{\mathrm{e}}\right)^n=\lim_{n\to\infty}\Big(1-\frac{1}{2n}+o\Big(\frac{1}{n}\Big)\Big)^n=\frac{1}{\sqrt{\mathrm{e}}}$$

□

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Problem 3 *Give an example of a sequence of continuous functions on* $\mathbb{R}$ *converging pointwise to* 0 *which is not uniformly convergent on any nonempty open set.*

Solution Define for $n \in \mathbb{N}$ a continuous function g_n by $g_n(t) = 0$ for $t \le 0$ or $t \ge \frac{2}{n}$, $g_n\left(\frac{1}{n}\right) = 1$ and g_n is linear on $\left[0, \frac{1}{n}\right]$, $\left[\frac{1}{n}, \frac{1}{2n}\right]$. Put

$$f_n(t) = \sum_{k=0}^{\infty} 2^{-k} g_n(t - r_k)\,, \tag{1}$$

where r_k is an enumeration of rationals. By the Weierstrass theorem, the f_n are continuous. To prove that f_n tends pointwise to 0, fix $t \in \mathbb{R}$ and $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that $\sum\limits_{k=m}^{\infty} 2^{-k} < \frac{\varepsilon}{2}$. By (1) and the definition of g_n, we can choose $n \in \mathbb{N}$ such that for $l \ge n$

$$\sum_{k=0}^{m-1} 2^{-k} g_l(t - r_k) < \frac{\varepsilon}{2},$$

which proves our claim. Now suppose that there exists an open, nonempty interval I such that $f_n \to 0$ uniformly on I. Choose $k_0 \in \mathbb{N}$ with $r_{k_0} \in I$. Put $t_n = r_{k_0} + \frac{1}{n}$. Then for n sufficiently large,

$$\sup_{t \in I} |f_n(t)| \ge f_n(t_n) \ge 2^{-k_0} g_n(t_n - r_{k_0}) = 2^{-k_0} g_n \frac{1}{n} = 2^{-k_0};$$

a contradiction. $\square$

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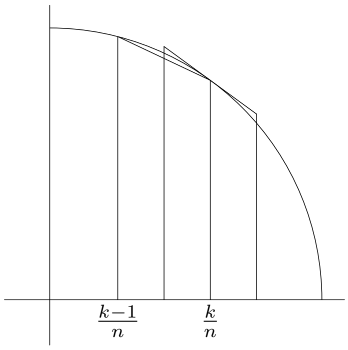
Problem 4-M *Prove the inequality*

$$\frac{n\pi}{4} - \frac{1}{\sqrt{8n}} \leq \frac{1}{2} + \sum_{k=1}^{n-1} \sqrt{1 - \frac{k^2}{n^2}} \leq \frac{n\pi}{4} \tag{1}$$

for every integer $n \geq 2$.

Solution We obtain the inequalities by approximating the area of the first quarter of the unit circle with the sum of the areas of inscribed and tangent trapezoids (see the figure).



Obviously the sum of the area of the inscribed trapezoids with vertices

$$\Big(\frac{k-1}{n}, 0\Big), \Big(\frac{k}{n}, 0\Big), \Big(\frac{k}{n}, \sqrt{1 - \frac{k^2}{n^2}}\Big), \Big(\frac{k-1}{n}, \sqrt{1 - \frac{(k-1)^2}{n^2}}\Big)$$

(from $k = 1$ to $k = n$) estimates $\frac{\pi}{4}$ (the area of the quarter of the unit circle) from below. The areas of such a trapezoid is

$$\frac{1}{n} \frac{\sqrt{1 - \frac{(k-1)^2}{n^2}} + \sqrt{1 - \frac{k^2}{n^2}}}{2},$$

thus after summation we obtain the inequality

$$\frac{1}{2n} + \frac{1}{n} \sum_{k=1}^{n-1} \sqrt{1 - \frac{k^2}{n^2}} \leq \frac{\pi}{4}, \tag{2}$$

which immediatelly implies the second inequality in (1).

The set of the points of the unit circle with positive ordinate and with abscissa between $\frac{2k-1}{2n}$, and $\frac{2k+1}{2n}$ is a subset of the trapezoid determined by the horizontal axis, the vertical lines with abscissas $\frac{2k-1}{2n}$ and $\frac{2k+1}{2n}$, and the tangent line of the unit circle at the point $\Big(\frac{k}{n}, \sqrt{1 - \frac{k^2}{n^2}}\Big)$. The area of this trapezoid is simply $\frac{1}{n}\sqrt{1 - \frac{k^2}{n^2}}$. We still have to cover the points of the first quarter of the unit circle with abscissas less than $\frac{1}{2n}$ or greater than $\frac{2n-1}{2n}$. The former part is simply covered by a rectangle of area $\frac{1}{2n}$, while the latter part is again covered by a tangent trapezoid of area T_n, which satisfies

$$T_n = \frac{1}{2n}\sqrt{1 - \Big(\frac{4n-1}{4n}\Big)^2} = \frac{1}{2n}\sqrt{\frac{8n-1}{16n^2}} \leq \frac{1}{2n} \frac{1}{\sqrt{2n}}.$$

The sum of these areas estimates the area of the quarter of the unit circle from above, therefore we have

$$\frac{\pi}{4} \leq \frac{1}{2n} + \frac{1}{n} \sum_{k=1}^{n-1} \sqrt{1 - \frac{k^2}{n^2}} + \frac{1}{2n} \frac{1}{\sqrt{2n}}. \tag{3}$$

Obviously inequality (3) is equivalent to the first inequality in (1). $\square$

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Problem 4-I *Prove that there exists a program in standard Pascal which prints out its own ascii code. No disc operations are permitted.*

Solution Let $\mathfrak{g}$ be a computable Gödel numbering over pascal programs. One can assume that every number is used in this numbering, i.e. every number is $\mathfrak{g}(K)$ for some program K. Let K_n be the program whose Gödel number is n and let $\mathfrak{k}(\sigma)$ be the following program, $\sigma \in \mathbb{N}$:

```
Program K ;
  const N = Δσ ;
  procedure PrintOut ( P : integer ) ;
    var ... end ; { PrintOut }
begin
  PrintOut ( N ) ;
end .
```

where Δ_σ is the decimal representation of σ, and `PrintOut` is the procedure which given a number n prints the program whose Gödel number is n. We will show that there exists σ satisfying

$$\mathfrak{g}(\mathfrak{k}(\sigma)). \tag{1}$$

Indeed, let

$$\Pi \overset{Df}{=} \langle \lambda s : \mathfrak{g}(\mathfrak{k}(s)) \rangle$$

and let

$$\delta \overset{Df}{=} \langle \lambda s : \text{ the output of } K_s \text{ running with } s \text{ as the input} \rangle.$$

Both the functions Π and δ are computable, hence the function $\Pi \circ \delta$ is computable. Let L be a program computing the latter and let M be the Gödel number of L. Then $\sigma \overset{Df}{=} \delta(M)$ satisfies (1). Indeed, we have:

$$\begin{aligned} \sigma = \delta(M) &= \langle \text{the value of the program } K_M \text{ on data } M \rangle = \\ &= \langle \text{the value of the program } L \text{ on data } M \rangle = (\Pi \circ \delta)(M) = \\ &= \Pi(\delta(M)) = \Pi(\sigma) = \mathfrak{g}(\mathfrak{k}(\sigma)). \end{aligned}$$

So the desired program is $\mathfrak{k}(\sigma)$. To see that, let us notice that $\mathfrak{k}(\sigma)$ prints K_σ, i.e. it prints $K_{\mathfrak{g}(\mathfrak{k}(\sigma))} = \mathfrak{k}(\sigma)$. Hence it prints itself. $\square$

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Category II

Problem 1 *Let H be a complex Hilbert space. Let $T\colon H \to H$ be a bounded linear operator such that $|(Tx, x)| \le \|x\|^2$ for each $x \in H$. Assume that $\mu \in \mathbb{C}$, $|\mu| = 1$, is an eigenvalue with the corresponding eigenspace $E = \{\phi \in H : T\phi = \mu\phi\}$. Prove that the orthogonal complement $E^\perp = \{x \in H : \forall\phi \in E : (x, \phi) = 0\}$ of E is T-invariant, i.e., $T(E^\perp) \subseteq E^\perp$.*

Solution Suppose $x \in E^\perp, \|x\| = 1$. This means that $(x, \phi) = 0$ for each $\phi \in E$. Then for any $t \in \mathbb{C}$ and $\phi \in E, \|\phi\| = 1$, we have

$$\begin{aligned} 1 + |t|^2 = \|\phi + tx\|^2 &\ge |T(\phi + tx), \phi + tx| \\ &= |(T\phi, \phi) + t(Tx, \phi) + \bar{t}(T\phi, x) + |t|^2(Tx, x)| \\ &= |\mu + t(Tx, \phi) + |t|^2(Tx, x)| \end{aligned}$$

because $(T\phi, x) = \mu(\phi, x) = 0$. If $(Tx, x) \ne 0$ we can take

$$t = \frac{|(T\phi, x)|^2}{|(Tx, x)|^2} \frac{(Tx, x)}{(Tx, \phi)} \ne 0$$

to obtain $1 + |t|^2 \le |\mu + 0| \le 1$, a contradiction. In the case $(Tx, x) = 0$ one can take $t = -\mu/(Tx, \phi)$ yielding a contradiction again. This implies that $(Tx, \phi) = 0$ for any $\phi \in E$, i.e. $Tx \in E^\perp$. $\square$

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Problem 2 *Decide, whether there is a member in the arithmetic progression* $\{a_n\}_{n=1}^{\infty}$ *with first member* $a_1 = 1998$ *and common difference* $d = 131$ *which is a palindrome (a palindrome is a number whose decadic expression is symmetric, e.g.* 7, 33, 43334, 2135312 *and so on).*

Solution We know that $10^{131-1} \equiv 1 \pmod{131}$ (because 131 is a prime). We shall construct the following number which shall be a member of our arithmetic progression:

$$A = 8991 \quad \underbrace{0\ldots0}_{\text{126-times}} \quad 1 \quad \underbrace{0\ldots0}_{\text{130-times}} \quad 1 \quad \underbrace{0\ldots0}_{\text{130-times}} \quad 1\ldots\ldots\ldots1 \quad \underbrace{0\ldots0}_{\text{130-times}} \quad 1 \quad \underbrace{0\ldots0}_{\text{126-times}} \quad 1998$$

(there is a suitable number of ones inside of the number - we shall show precisely later). We can express the number in a more precise manner:

$$A = 8991 \cdot 10^{126} \cdot 10^{130 \cdot r} + \sum_{i=1}^{r} 10^{130 \cdot i} + 1998$$

Now we can find out for which walues of r is the number a member of our arithmetic progression (A is a member of the sequence if 131 divides $A - 1998$):

$$A - 1998 = 8991 \cdot 10^{126} \cdot 10^{130 \cdot r} + \sum_{i=1}^{r} 10^{130 \cdot i} \equiv t \cdot 1^r + \sum_{i=1}^{r} 1^i \equiv t + r \pmod{131},$$

where t is the remainder on dividing $8991 \cdot 10^{126}$ by 131. Now if $t + r$ is a multiple of 131 then A will be a member of the sequence. But there are infinitely many such values of r which solve the problem. $\square$

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Problem 3 *Show that the roots of the real polynomial*

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n,$$

where $0 < a_0 < \cdots < a_n$, *satisfy* $|z| > 1$.

Solution Observe that for any $z \notin (0,1]$ on the unit disc

$$\begin{aligned}
|(1-z)P(z)| &= |\sum_{j=0}^{n} a_j (z^{n-j} - z^{n+1-j})| \\
&> |a_n| - (\sum_{j=0}^{n-1} |a_j - a_{j+1}| \cdot |z^{n-j}|) \\
&\geq |a_n| - \sum_{j=0}^{n-1} |a_j - a_{j+1}| \\
&= a_n - (\sum_{j=0}^{n-1} (a_{j+1} - a_j) + a_0) = 0.
\end{aligned}$$

The strict inequality follows from the fact, that if

$$|\sum_{j=0}^{n-1} (a_j - a_{j+1}) z^{n-j} - a_0 z^{n+1}| = \sum_{j=0}^{n-1} |(a_j - a_{j+1}) z^{n-j}| + |a_0 z^{n+1}|$$

then

$$|(a_0 - a_1) z^n| + |(a_1 - a_2) z^{n-1}| = |(a_0 - a_1) z^n + (a_1 - a_2) z^{n-1}|$$

and

$$|(a_0 - a_1) z| + |(a_1 - a_2)| = |(a_0 - a_1) z + (a_1 - a_2)|.$$

Consequently, $z > 0$. Since $|z| \leq 1$, $z \in (0,1]$; a contradiction.

Hence for $z \neq 1$, $P(z) \neq 0$. Since $P(z) > 0$ for $z \in (0,1]$, the result is proved. $\square$

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Problem 4-M *A function $f\colon \mathbb{R} \to \mathbb{R}$ has the property that for every $x, y \in \mathbb{R}$ there exists a real number t (depending on x and y) such that $0 < t < 1$ and*

$$f\bigl(tx + (1-t)y\bigr) = tf(x) + (1-t)f(y)\,. \tag{1}$$

Does this imply that

$$f\Bigl(\frac{x+y}{2}\Bigr) = \frac{f(x)+f(y)}{2} \tag{2}$$

for every $x, y \in \mathbb{R}$?

Solution As is justified by the following counterexample, the answer is negative. Since the sets $\mathbb{R} \setminus \{0\}$ and $\mathbb{R}^2$ have the same cardinality, there exists a surjective mapping $\gamma \mapsto (a_\gamma, b_\gamma)$ from $\mathbb{R} \setminus \{0\}$ onto $\mathbb{R}^2$. Let us define

$$\phi_\gamma(x) = a_\gamma x + b_\gamma \quad (x \in \mathbb{R}) \quad \text{for every } \gamma \in \mathbb{R} \setminus \{0\} \text{ and } \phi_0(x) = |x| \quad (x \in \mathbb{R}).$$

There also exists a family $\{A_\gamma : \gamma \in \mathbb{R}\}$ of pairwise disjoint sets such that A_γ is a dense subset of $\mathbb{R}$ for every $\gamma \in \mathbb{R} \setminus \{0\}$ and $A_0 = \{-1, 0, 1\}$. Now we can define

$$f(x) = \begin{cases} \phi_\gamma(x) & \text{if } x \in A_\gamma\,,\ \gamma \in \mathbb{R}\,, \\ 0 & \text{if } x \in \mathbb{R} \setminus \bigcup_{\gamma \in \mathbb{R}} A_\gamma\,. \end{cases} \tag{3}$$

For every $x, y \in \mathbb{R}$ with $x \neq y$ there exists $\beta \in \mathbb{R} \setminus \{0\}$ such that

$$\frac{f(y)-f(x)}{y-x} = a_\beta \quad \text{and} \quad \frac{yf(x)-xf(y)}{y-x} = b_\beta\,.$$

Since A_β is a dense set, there exists $w \in A_\beta$ strictly between x and y. Then $f(u) = \phi_\beta(u)$ for $u \in \{x, y, w\}$, hence (1) is satisfied with $t = \frac{y-u}{y-x}$. On the other hand

$$f\Bigl(\frac{-1+1}{2}\Bigr) = f(0) = \phi_0(0) = 0 \neq 1 = \frac{\phi_0(-1)+\phi_0(1)}{2} = \frac{f(-1)+f(1)}{2}\,.$$

□

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Problem 4-I *Let us consider the first order language L with just one 3-ary predicate $PLUS$; hence (well-formed) formulas of L contain symbols for variables, logical connectives, quantifiers, brackets, and the predicate symbol $PLUS$: $(\exists x_1)(\forall x_2) : (Plus(x_2, x_1, x_2) \wedge (\forall x_3) : \neg Plus(x_1, x_3, x_3))$ is example of such formula). Recall that a formula is* closed *iff each variable symbol occurs within the scope of a quantifier.*

Show that there is an algorithm which decides whether or not a given closed formula of L is true for the set $\mathcal{N}$ of natural numbers ($\{0, 1, 2, \dots\}$) where $Plus(x, y, z)$ is interpreted as $x + y = z$.

Solution We start by showing that there is a finite automaton recognizing the language described in the Hint, let us denote it L_1. But this is obvious when we consider the reverse image $(L_1)^R$ (the respective automaton just checks correctness of the binary addition), and recall that the class Reg of languages recognizable by finite automata is closed under the reverse operation.

The desired algorithm works as follows:

The given formula is transformed into the prenex form

$$(Q_1x_1)(Q_2x_2)\dots(Q_nx_n)\mathcal{F}(x_1, x_2, \dots, x_n)$$

(where Q_i is either $\exists$ or $\forall$).

By generalizing the above construction of finite automaton, we construct an automaton A_n which accepts exactly those words which represent n-tuples $a_1, a_2, \dots, a_n$ s.t. $\mathcal{F}(a_1, a_2, \dots, a_n)$ is true. This can be done by using the fact that Reg is closed under union and complement (as well as intersection).

When $Q_n = \exists$ we can, from A_n, construct A_{n-1} which accepts exactly the words representing $(n-1)$-tuples $a_1, a_2, \dots, a_{n-1}$ s.t. $(\exists x_n)\mathcal{F}(a_1, a_2, \dots, a_{n-1}, x_n)$ is true as follows: A_{n-1} will simulate A_n by nondeterministically guessing the appropriate bits of the n-th number; it can also start by a sequence of ε-moves (not reading input) which capture the possibility that the n-th number has more significant bits than the other numbers. Of course, A_{n-1} can be made deterministic by a standard construction.

If $Q_n = \forall$ we realize that $(\forall x_n)\mathcal{F}(a_1, a_2, \dots, a_{n-1}x_n)$ is equivalent to

$$\neg(\exists x_n)\neg\mathcal{F}(a_1, a_2, \dots, a_{n-1}, x_n)$$

and proceed similarly (Reg is closed under complement).

Thus we successively construct automata $A_n, A_{n-1}, \dots, A_1, A_0$. In the end, it remains to check if A_0 accepts the empty word. $\square$