The 8th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 1st April 1998
Category II

Problem 1. Let H be a complex Hilbert space. Let $T\colon H \to H$ be a bounded linear operator such that $\bigl|(Tx, x)\bigr| \le \|x\|^2$ for each $x \in H$. Assume that $\mu \in \mathbb{C}$, $|\mu| = 1$, is an eigenvalue with the corresponding eigenspace $E = \{\, \phi \in H : T\phi = \mu\phi \,\}$. Prove that the orthogonal complement $E^{\perp} = \{\, x \in H : \forall \phi \in E\colon (x, \phi) = 0 \,\}$ of E is T-invariant, i.e., $T(E^{\perp}) \subseteq E^{\perp}$. [15 points]

Problem 2. Decide whether there is a member in the arithmetic sequence $\{a_n\}_{n=1}^{\infty}$ whose first member is $a_1 = 1998$ and the common difference $d = 131$ which is a palindrome (palindrome is a number such that its decimal expansion is symmetric, e.g., 7, 33, 433334, 2135312 and so on). [25 points]

Problem 3. Show that all complex roots of the polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$, where $0 < a_0 < \cdots < a_n$, satisfy $|z| > 1$. [25 points]

Problem 4-M. A function $f\colon \mathbb{R} \to \mathbb{R}$ has the property that for every $x, y \in \mathbb{R}$ there exists a real number t (depending on x and y) such that $0 < t < 1$ and

$$f\bigl(tx + (1-t)y\bigr) = tf(x) + (1-t)f(y)\,.$$

Does it imply that

$$f\Bigl(\frac{x+y}{2}\Bigr) = \frac{f(x) + f(y)}{2}$$

for every $x, y \in \mathbb{R}$? [35 points]

Problem 4-I. Let us consider a first-order language L with a ternary predicate Plus. Hence (well-formed) formulas of L are built of symbols for variables, logical connectives, quantifiers, brackets, and the predicate symbol Plus.

$$(\exists x_1)(\forall x_2)\colon \bigl(\mathrm{Plus}(x_2, x_1, x_2) \wedge (\forall x_3)\colon \neg\mathrm{Plus}(x_1, x_3, x_3)\bigr)$$

is an example of such a formula. Recall that a formula is *closed* iff each variable symbol occurs within the scope of a quantifier.

Show that there exists an algorithm which decides whether or not a given closed formula of L is true for the set $\mathbb{N}$ of natural numbers ($\{0, 1, 2, \ldots\}$) where $\mathrm{Plus}(x, y, z)$ is interpreted as $x + y = z$.
[35 points]