The 7th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 9th April 1997
Category I

Problem 1 *Let a be an odd positive integer. Prove that if $d \mid (a^2+2)$ then $d \equiv 1 \pmod 8$ or $d \equiv 3 \pmod 8$.*

Solution If $p \mid (a^2+2)$ and p is the prime, then -2 is a quadratic residue modulo p. It follows that for the Legendre symbol $\left(\frac{-2}{p}\right) = 1$, where $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre symbol. Using the properties of Legendre symbol we obtain

$$\left(\frac{-2}{p}\right) = \left(\frac{-1}{p}\right) \cdot \left(\frac{2}{p}\right) = (-1)^{\frac{p-1}{2}+\frac{p^2-1}{8}}.$$

Thus the number $\frac{p-1}{2}+\frac{p^2-1}{8}$ is even and so

$$p^2 + 4p \equiv 5 \pmod{16}. \tag{1}$$

On the other hand p is an odd prime, thus p is of the form $8k+1, 8k+3, 8k+5$ or $8k+7$. This and (1) yield that the prime p is of the form $8k+1$ or $8k+3$. The product of an even number of numbers of the form $8k+3$ is a number b fulfilling $b \equiv 1 \pmod 8$, and the product of an odd number of numbers of the form $8k+3$ is c fulfilling $c \equiv 3 \pmod 8$. The product e of numbers of the form $8k+1$ satisfies $e \equiv 1 \pmod 8$. This yields the assertion. □

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Problem 2 *Let $\alpha \in (0,1]$ be a given real number and let the real sequence $\{a_n\}_{n=1}^{\infty}$ satisfy the inequality*

$$a_{n+1} \le \alpha a_n + (1-\alpha)a_{n-1} \quad \textit{for } n = 2,3,\ldots$$

If $\{a_n\}$ is bounded prove that then it must be convergent.

Solution Since (a_n) is bounded there exists both $\liminf a_n = l$ and $\limsup a_n = L$. We shall prove that $L = l$. Let suppose the contrary $L > l$. By the definition of lim sup and lim inf we know that there exist subsequencies (a_{n_k}) and (a_{m_k}) of (a_n) which converge to l and L.

Since $L = \limsup a_n$ we have:

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \ge n_0) a_n < L + \varepsilon.$$

On the other hand

$$(\forall \varepsilon > 0)(\exists n_1 \in \mathbb{N})(\forall n \ge n_1) l - \varepsilon < a_n.$$

We choose m_{k_0} such that: $m_{k_0} > n_1, l - \varepsilon < a_{m_{k_0}} < l + \varepsilon$. By the inequality we obtain:

$$a_{m_{k_0+1}} \le (1-\alpha)a_{m_{k_0}} + \alpha a_{m_{k_0-1}} < (1-\alpha)(l+\varepsilon) + \alpha(L+\varepsilon) = \varepsilon + (1-\alpha)l + \alpha L$$

and

$$a_{m_{k_0+2}} \le (1-\alpha)a_{m_{k_0+1}} + \alpha a_{m_{k_0}} < (1-\alpha)(\varepsilon + (1-\alpha)l + \alpha L) + \alpha(l+\varepsilon) == \varepsilon + ((1-\alpha)^2 + \alpha)l + (1-\alpha)\alpha L.$$

It is not difficult to demonstrate that we can choose ε such that:

$$\varepsilon + (1-\alpha)l + \alpha L < L - \frac{\varepsilon}{2},$$
$$\varepsilon + (1 - \alpha + \alpha^2)l + (\alpha - \alpha^2)L < L - \frac{\varepsilon}{2},$$

using the functions: $f(x) = \varepsilon + (1-x)l + xL$ and $g(x) = \varepsilon + (1 - x - x^2)l + (x - x^2)L$, $x \in [0,1]$.

From that we have: $a_{m_{k_0+1}} \le L - \frac{\varepsilon}{2}, a_{m_{k_0+2}} \le L - \frac{\varepsilon}{2}$. From that we obtain:

$$a_{m_{k_0+3}} \le (1-\alpha)a_{m_{k_0+1}} + a_{m_{k_0}} < (1-\alpha)\Big(L - \frac{\varepsilon}{2}\Big) + \alpha\Big(L - \frac{\varepsilon}{2}\Big) = L - \frac{\varepsilon}{2},$$

therefore by induction we get:

$$a_n < L - \frac{\varepsilon}{2}, \quad \forall n \ge m_{k_0+1}.$$

The last formula yields:

$$\limsup a_n \le L - \frac{\varepsilon}{2} < L,$$

which is a contradiction. Thus $L = l$. □

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Problem 3 *Let $c_1, c_2, \dots, c_n$ be real numbers such that*

$$c_1^k + c_2^k + \cdots + c_n^k > 0 \quad \text{for } k = 1, 2, \dots \tag{1}$$

Let us put

$$f(x) = \frac{1}{(1 - c_1 x)(1 - c_2 x) \dots (1 - c_n x)}.$$

Show that $f^{(k)}(0) > 0$ for all $k = 1, 2, \dots$.

Solution Put $g(x) := \log f(x)$. Then

$$g(x) = -\sum_{j=1}^{n} \log(1 - c_j x) \quad \text{and} \quad g^{(k)}(x) = \sum_{j=1}^{n} \frac{(k-1)! c_j^k}{(1 - c_j x)^k}.$$

Assumption (1) and the above result give

$$g^{(k)}(0) = c_1^k + c_2^k + \cdots + c_n^k > 0.$$

Now observe that if all the derivatives of g at the origin are positive, then e^g has the same property. For this show by induction that

$$(\mathrm{e}^g)^{(k)} = \mathrm{e}^g \cdot S\,,$$

where S is a finite sum of terms of the form

$$a(g^{(l_1)})^{m_1}(g^{(l_2)})^{m_2} \dots (g^{(l_r)})^{m_r},$$

where a, l_j and m_j are positive integers. For instance:

$$\begin{aligned}
(\mathrm{e}^{g(x)})' &= \mathrm{e}^{g(x)} g'(x) \\
(\mathrm{e}^{g(x)})'' &= \mathrm{e}^{g(x)}((g'(x))^2 + g''(x)) \\
(\mathrm{e}^{g(x)})''' &= \mathrm{e}^{g(x)}((g'(x))^3 + 3g'(x)g''(x) + g'''(x)) \\
(\mathrm{e}^{g(x)})^{(4)} &= \mathrm{e}^{g(x)}((g'(x))^4 + 6(g'(x))^2 g''(x) + 3(g''(x))^2 + 4g'(x)g'''(x) + g^{(4)}(x))
\end{aligned}$$

□

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Problem 4-M *Find all real numbers $a > 0$ for which the series*

$$\sum_{n=1}^{\infty} \frac{a^{f(n)}}{n^2}$$

is convergent, where $f(n)$ denotes the number of zeros in the decimal expansion of n.

Solution For $n = 0, 1, \ldots$ let us consider k's fulfilling the inequality:

$$10^n \le k < 10^{n+1}. \tag{1}$$

For a fixed $0 \le j \le n$ there are

$$\binom{n}{j} 9^{n-j+1} \quad k\text{'s fulfilling (1) such that } f(k) = j.$$

Consequently,

$$\frac{9(9+a)^n}{10^{2n+2}} = \frac{9}{10^{2n+2}} \left(\sum_{j=0}^{n} \binom{n}{j} a^j 9^{n-j} \right) \le \sum_{k=10^n}^{10^{n+1}-1} \frac{a^{f(k)}}{k^2} \le \frac{9}{10^{2n}} \left(\sum_{j=0}^{n} \binom{n}{j} a^j 9^{n-j} \right) = \frac{9(9+a)^n}{10^{2n}}.$$

This implies that

$$\frac{9}{100} \sum_{n=0}^{\infty} \left(\frac{9+a}{100} \right)^n \le \sum_{n=1}^{\infty} \frac{a^{f(n)}}{n^2} \le 9 \sum_{n=0}^{\infty} \left(\frac{9+a}{100} \right)^n.$$

From the last inequalities we conclude that our series is convergent exactly when

$$\frac{9+a}{100} < 1$$

or

$$0 < a < 91.$$

□

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Category I

Problem 4-I *Let us have declared:*

```
const N_MAX = 255;
type
   tR = array[1..N_MAX] of real;
   tN = array[1..N_MAX] of integer;
```

and function random without parameters which returns real random values distributed uniformly in $[0, 1)$.

You need to choose K *integer numbers* $(1 < K < N, N\_MAX \geq N)$ *without repetitions under the condition that the probability of the choice of a number* i *equals a given* P_i, $\sum_{i=1}^{N} P_i = 1$.

Write the procedure in Pascal that returns such K *integer numbers in the first* K *elements of the vector of* tN *type. Input parameters of the procedure are* K, N *and the vector of* P_i.

Solution

□

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Category II

Problem 1 *Decide whether it is possible to cover a 3-dimensional Euclidean space with lines which are pairwise skew (i.e. not coplanar).*

Solution There is (for example) the following covering:

$$\text{lines } p_{a,b} = \{A_{a,b} = [a, b, 0]; s_{a,b} = (-b, a, 1)\},$$

where $A_{a,b}$ is point and $s_{a,b}$ is vector of $p_{a,b}$. We show that

1. if $[a, b] \neq [c, d]$ then $p_{a,b} \cap p_{c,d} = \emptyset$; and
2. for each $X = [x, y, z]$ there is $[a, b]$ such that $X \in p_{a,b}$.

1. Let $[a, b] \neq [c, d]$ and $p_{a,b} \cap p_{c,d} = \emptyset$. Then (from the parametric expression)

$$\begin{aligned} a - tb &= c - rd \\ b + ta &= d + rc \\ t &= r \end{aligned}$$

   for some real t and r. From the third equality $t = r$ and from the first and the second we have $a-c = t(b-d)$ and $b - d = -t(a - c)$. By linear combination (first times $(a - c)$ plus second times $(b - d)$) we have

$$(a - c)^2 + (b - d)^2 = 0.$$

   But this contradicts $[a, b] \neq [c, d]$.

2. For $X = [x, y, z]$ put $a = \frac{x+yz}{1+z^2}$; $b = \frac{y-xz}{1+z^2}$. Then $X = A_{a,b} + z \cdot s_{a,b}$.

This implies that it is possible to cover 3-dimensional Euclidean space with lines which are pairwise skew. □

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Problem 2 *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = 1$ for all $z \in \mathbb{C}$ with $|z| = 1$. Prove that there are $\theta \in \mathbb{R}, k \in \{0, 1, 2, \dots\}$ such that*

$$f(z) = \mathrm{e}^{\mathrm{i}\theta} z^k \quad \text{for any } z \in \mathbb{C}.$$

Solution We know that

$$|f(z)| = 1 \quad \text{for all } |z| = 1. \tag{1}$$

Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Let $\{z_1, \dots, z_n\}$ be zeros (counted with multiplicity) of the function f belonging to Δ. The number of zeros is finite because of the identity principle and (1).

Let us define the following function:

$$g(z) := \frac{f(z)}{\prod_{j=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z}}, \quad \text{for all } z \text{ with } z\bar{z}_j \neq 1.$$

From the choice of z_j we certainly have that g is holomorphic in some neighbourhood of $\bar{\Delta}$ (even in $\mathbb{C} \setminus \{1/\bar{z}_1, \dots, 1/\bar{z}_n\}$).

Because of (1) and the fact that

$$\left| \frac{z - z_j}{1 - \bar{z}_j z} \right| = 1 \quad \text{for all } |z| = 1$$

we have that

$$|g(z)| = 1 \quad \text{for all } |z| = 1. \tag{2}$$

Moreover,

$$g(z) \neq 0 \quad \text{for all } z \in \Delta. \tag{3}$$

In view of (2) and (3), the minimum and maximum principles applied to the mapping g imply that

$$|g| \equiv 1 \text{ on } \Delta,$$

which gives us that

$$g \equiv \mathrm{e}^{\mathrm{i}\theta} \quad \text{for some } \theta \in \mathbb{R}.$$

Therefore,

$$f(z) = \mathrm{e}^{\mathrm{i}\theta} \prod_{j=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z}, \quad z \in \mathbb{C}.$$

And the last inequality is possible iff $z_1 = \cdots = z_n = 0$. $\square$

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Problem 3 *Let $u \in C^2(\bar{D}), u = 0$ on ∂D, where D is an open unit ball in $\mathbb{R}^3$. Prove that the following inequality*

$$\int_D |\text{grad } u|^2 \, \mathrm{d}V \le \varepsilon \int_D (\Delta u)^2 \, \mathrm{d}V + \frac{1}{4\varepsilon} \int_D u^2 \, \mathrm{d}V$$

holds for all $\varepsilon > 0$.

Solution Since the inequality must hold for every $\varepsilon > 0$ it will hold too for those ε for which the function $g(\varepsilon) = \varepsilon \int_D (\Delta u)^2 \, \mathrm{d}V + \frac{1}{4\varepsilon} \int_D u^2 \, \mathrm{d}V$ takes a minimum, i.e., for $\varepsilon = \frac{1}{2}\sqrt{\frac{I_u}{I_l}}$ $(g'(\varepsilon) = I_l - \frac{1}{4\varepsilon^2} I_u = 0)$. That minimum is $g(\frac{1}{2}\sqrt{\frac{I_u}{I_l}}) = \sqrt{I_u I_l}$. Therefore it suffices to show that

$$\int_D |\text{grad } u|^2 \, \mathrm{d}V \le \sqrt{\int_D (\Delta u)^2 \, \mathrm{d}V \int_D u^2 \, \mathrm{d}V}\,.$$

By placing $(P, Q, R) = u(u_x, u_y, u_z)$ in the Gauss-Ostrogradski formula we get:

$$\int_D |\text{grad } u|^2 \, \mathrm{d}V + \int_D u \Delta u \, \mathrm{d}V = -\int_{\partial D} u \frac{\partial u}{\partial n} \, \mathrm{d}S = 0,$$

where n is inner normal. From this by the Cauchy-Schwartz inequality and the conditions of that statement we obtain

$$\int_D |\text{grad } u|^2 \, \mathrm{d}V = -\int_D u \Delta u \, \mathrm{d}V \le \sqrt{\int_D (\Delta u)^2 \, \mathrm{d}V \int_D u^2 \, \mathrm{d}V}\,.$$

□

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Problem 4-M *Prove that*

$$\sum_{n=1}^{\infty}\frac{n^2}{(7n)!}=\frac{1}{7^3}\sum_{k=1}^{2}\sum_{j=0}^{6}\mathrm{e}^{\cos\left(\frac{2\pi j}{7}\right)}\cdot\cos\left(\sin\left(\frac{2\pi j}{7}\right)+\left(\frac{2\pi jk}{7}\right)\right).$$

Solution

$$\sum_{n=1}^{\infty}\frac{n^2}{(7n)!}=\frac{1}{49}\sum_{n=1}^{\infty}\frac{7n(7n-1)+7n}{(7n)!}=\sum_{k=1}^{2}\sum_{n=1}^{\infty}\frac{1}{(7n-k)!}$$

We have

$$A=\sum_{k=1}^{2}\sum_{j=0}^{6}\mathrm{e}^{\mathrm{e}^{\frac{2\pi \mathrm{i}j}{7}}+\frac{2\pi \mathrm{i}jk}{7}}=\sum_{k=1}^{2}\sum_{j=0}^{6}\mathrm{e}^{\frac{2\pi \mathrm{i}jk}{7}}\mathrm{e}^{\mathrm{e}^{\frac{2\pi \mathrm{i}j}{7}}}=\sum_{k=1}^{2}\sum_{j=0}^{6}\mathrm{e}^{\frac{2\pi \mathrm{i}jk}{7}}\sum_{n=0}^{\infty}\frac{\mathrm{e}^{\frac{2\pi \mathrm{i}jk}{7}}}{n!}=\sum_{k=1}^{2}\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{j=0}^{6}\mathrm{e}^{\frac{2\pi \mathrm{i}j(n+k)}{7}}.$$

If $7 \mid N$, then

$$\sum_{j=0}^{6}\mathrm{e}^{\frac{2\pi \mathrm{i}jN}{7}}=7.$$

If $7 \nmid N$, then

$$\sum_{j=0}^{6}\mathrm{e}^{\frac{2\pi \mathrm{i}jN}{7}}=\frac{\mathrm{e}^{2\pi \mathrm{i}N}-1}{\mathrm{e}^{\frac{2\pi \mathrm{i}jN}{7}}-1}=0.$$

It follows that

$$A=7\sum_{k=1}^{2}\sum_{n=1}^{\infty}\frac{1}{(7n-k)!}.$$

Thus

$$\sum_{n=1}^{\infty}\frac{n^2}{(7n)!}=\frac{1}{7^3}\sum_{k=1}^{2}\sum_{j=0}^{6}\mathrm{e}^{\mathrm{e}^{\frac{2\pi \mathrm{i}j}{7}}+\frac{2\pi \mathrm{i}jk}{7}}=\frac{1}{7^3}\sum_{k=1}^{2}\sum_{j=0}^{6}\mathrm{e}^{\cos(\frac{2\pi j}{7})+\mathrm{i}(\sin(\frac{2\pi j}{7})+(\frac{2\pi jk}{7}))}=$$

$$=\frac{1}{7^3}\sum_{k=1}^{2}\sum_{j=0}^{6}\mathrm{e}^{\cos(\frac{2\pi j}{7})}\cdot\cos\left(\sin\left(\frac{2\pi j}{7}\right)+\left(\frac{2\pi jk}{7}\right)\right).$$

□

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Problem 4-I *Problem Div_3 is specified as follows:*

Input: *any program P,*

Output: *a finite set $D(P)$ of strings of 0's and 1's,*

where it holds that the program P solves problem Div_3 iff it outputs the correct answers for inputs from $D(P)$.

Theorem *For any program $TRANSF$ which transforms programs in some way (i.e. for any given program P it constructs some program P', denoted by $P' = TRANSF(P)$) there is a program P_0 whose input/output behaviour is not changed by the transformation (i.e. P_0 and $TRANSF(P_0)$ yield the same outputs for the same inputs).*

Solution Suppose there is some GEN-TEST-DATA with the property described above. Now construct a program $TRANSF$ which works as follows:

When given a program P, it constructs $D(P)$ by using GEN-TEST-DATA and then constructs P' whose behaviour can be described as follows:

```
s:= input;
if s ∈ D(P);
then output YES or NO according to whether or not s is
     the binary code of a number divisible by 3;
else output YES;
```

Due to the Recursion Theorem there is a program P_0 s.t. its input/output behaviour is the same as the behaviour of $TRANSF(P_0)$, and it can be described as follows:

```
s:= input;
if s ∈ D(P_0);
then output YES or NO according to whether or not s is
     the binary code of a~number divisible by 3;
else output YES;
```

Such a program P_0 obviously violates the condition supposed for GEN-TEST-DATA.

Hence we have to conclude that there is no desired program GEN-TEST-DATA. □