The 5$^{\text{th}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 25$^{\text{th}}$ – 26$^{\text{th}}$ April 1995
Category I

Problem 1 *Discuss the solvability of the equations*

$$\lambda x + y + z = a$$
$$x + \lambda y + z = b$$
$$x + y + \lambda z = c$$

for all numbers $\lambda, a, b, c \in \mathbb{R}$.

Solution The characteristic polynomial of this system is $(\lambda-1)(\lambda-1)(\lambda+2)$. Hence we have three possibilities of solution:

1. For $\lambda = 1$ we obtain that if $a = b = c$ then the system has infinity many solutions in the form $x = a - u - v, y = u, z = v$, where u, v are parameters. Otherwise the system has no solution.

2. For $\lambda = -2$ we get if $a + b + c = 0$ the system has infinitely many solutions. Otherwise the system has no solution.

3. For $\lambda \in \mathbb{R} \setminus \{1, -2\}$ the system has just one solution.

$\square$

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Problem 2 *Let $f(x)$ be an even twice differentiable function such that $f''(0) \neq 0$. Prove that $f(x)$ has a local extremum at $x = 0$.*

Solution

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = f'(0)$$

and because $f(h) = f(-h)$ we have

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} + \lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \left(\frac{f(h) - f(0)}{h} + \frac{f(-h) - f(0)}{-h} \right) = 0$$
$$2f'(0) = 0\,.$$

Hence $f'(0) = 0$ and because $f''(0) \neq 0$, we obtain that $f(x)$ has an extremum at the point $x = 0$. $\square$

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Problem 3 *Let $f(x)$ and $g(x)$ be mutually inverse decreasing functions on the interval $(0, \infty)$. Can it hold that $f(x) > g(x)$ for all $x \in (0, \infty)$?*

Solution The answer is yes. For example, the functions $f(x) = -\frac{1}{2}x - 1$ and $g(x) = -2x - 2$ are mutually inverse functions and $f(x) > g(x)$ for all $x \in (0, \infty)$. $\square$

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Category II

Problem 1 *Prove that the systems of hyperbolas*

$$x^2 - y^2 = a \tag{1}$$
$$xy = b \tag{2}$$

are orthogonal.

Solution An arbitrary point lying on hyperbola (1) has the form $H_1 = [x, \sqrt{x^2 - a}]$ and $H_{\bar{1}} = [x, -\sqrt{x^2 - a}]$ and points lying on hyperbola (2) have the form $H_2 = [x, \frac{b}{x}]$. The tangent vectors to hyperbola (1) are $H_1' = (1, \frac{x}{\sqrt{x^2-a}})$ and $H_{\bar{1}}' = (1, \frac{-x}{\sqrt{x^2-a}})$ and the tangent vector to hyperbola (2) is $H_2' = (1, \frac{-b}{x^2}]$.

The point H is the point of intersection if and only if

$$H = [x, \sqrt{x^2 - a}] = [x, \frac{b}{x}] \Rightarrow \frac{b}{x} = \sqrt{x^2 - a} \tag{3}$$

or

$$H = [x, -\sqrt{x^2 - a}] = [x, \frac{b}{x}] \Rightarrow \frac{b}{x} = -\sqrt{x^2 - a}\,. \tag{4}$$

Hyperbolas are orthogonal at the intersection point if and only if the scalar product of the tangent vectors at this point is equal to 0. From (3) and (4) we have

$$H_1' \cdot H_2 = 1 - \frac{b}{x\sqrt{x^2 - a}} = 0$$

and

$$H_{\bar{1}}' \cdot H_2 = 1 - \frac{b}{x\sqrt{x^2 - a}} = 0\,.$$

So the systems of hyperbolas are orthogonal. □

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Problem 2 *Let $f = f_0 + f_1 z + f_2 z^2 + \ldots + f_{2n} z^{2n}$ and $f_k = f_{2n-k}$ for each k. Prove that $f(z) = z^n g(z + z^{-1})$, where g is a polynomial of degree n.*

Solution Let

$$\begin{aligned}
f_{2n} = f_0 &= a_n \binom{n}{0} \\
f_{2n-1} = f_1 &= a_{n-1} \binom{n-1}{0} \\
f_{2n-2} = f_2 &= a_n \binom{n}{1} + a_{n-2} \binom{n-2}{0} \\
f_{2n-3} = f_3 &= a_{n-1} \binom{n-1}{1} + a_{n-3} \binom{n-3}{0} \\
f_{2n-4} = f_4 &= a_n \binom{n}{2} + a_{n-2} \binom{n-2}{1} + a_n \binom{n-4}{0} \\
&\vdots \\
f_{2n-k} = f_k &= \begin{cases} \sum_{i=0}^{k/2} a_{n-2i} \binom{n-2i}{\frac{k}{2}-i} & \text{for } k \text{ even} \\ \sum_{i=0}^{\frac{k-1}{2}} a_{n-2i-1} \binom{n-2i-1}{\frac{k-1}{2}-i} & \text{for } k \text{ odd} \end{cases}
\end{aligned}$$

Then we get $f(z)$ in the form

$$\begin{aligned}
f(z) &= \sum_{k=0}^{n} a_k \Bigg(\binom{k}{0} (z^{n+k} + z^{n-k}) + \binom{k}{1} (z^{n+k-2} + z^{n-k+2}) + \ldots \\
&\quad + \begin{cases} \binom{k}{\frac{k}{2}} (z + z^{-1}) & \text{for } k \text{ even} \\ \binom{k}{\frac{k-1}{2}} (z + z^{-1}) & \text{for } k \text{ odd} \end{cases} \Bigg) \\
&= z^n \sum_{k=0}^{n} a_k \left(\binom{k}{0} (z^k + z^{-k}) + \binom{k}{1} (z^{k-2} + z^{2-k} + \ldots) \right) \\
&= z^n \sum_{k=0}^{n} a_k \left(\binom{k}{0} z^k + \binom{k}{1} z^{k-2} + \ldots + \binom{k}{k} z^{-k}) \right) \\
&= z^n \sum_{k=0}^{n} a_k (z + z^{-1})^k \\
&= z^n g(z + z^{-1}),
\end{aligned}$$

where g denotes the polynomial of degree n. $\square$

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Problem 3 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Do there exist continuous functions $g\colon \mathbb{R} \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)\sin x + h(x)\cos x$ holds for every $x \in \mathbb{R}$?*

Solution Let $g(x) = f(x)\sin x$ and $h(x) = f(x)\cos x$. Both functions $g(x)$ and $h(x)$ are continuous function and the condition

$$f(x) = g(x)\sin x + h(x)\cos x$$

holds. $\square$

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Problem 4 *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence such that* $x_1 = 25$, $x_n = \arctan x_{n-1}$. *Prove that this sequence has a limit and find it.*

Solution The sequence $\{x_n\}_{n=1}^{\infty}$ is a decreasing sequence because $0 < \frac{\arctan x}{x} < 1$ and $\{x_n\}_{n=1}^{\infty}$ is bounded by 0. So there exists a limit of this sequence. Let $\lim_{n\to\infty} x_n = L$. Then $L = \arctan L$, hence $L = 0$ and $\lim_{n\to\infty} x_n = 0$. $\square$