The 4^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 6^{th} April 1994
Category I

Problem 1 *Prove that an arbitrary integer can be written as a sum of five cube powers of integers.*

Solution For each n we have

$$6n = (n+1)^3 + (-n)^3 + (-n)^3 + (n-1)^3 \,.$$

Hence an arbitrary integer can be written in one of the following forms:

$$\begin{aligned}
6n+1 &= 6n + 1^3 \\
6n+2 &= 6(n-1) + 2^3 \\
6n+3 &= 6(n-4) + 3^3 \\
6n+4 &= 6(n+2) + (-2)^3 \\
6n+5 &= 6(n+1) + (-1)^3 \\
6n &= (n+1)^3 + (-n)^3 + (-n)^3 + (n-1)^3 + 0^3 \,.
\end{aligned}$$

□

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Problem 2 *Prove that for the roots x_1, x_2 of the polynomial*

$$x^2 - px - \frac{1}{2p^2},$$

where $p \in \mathbb{R}$ and $p \neq 0$, the following inequality holds:

$$x_1^4 + x_2^4 \geq 2 + \sqrt{2}.$$

Solution According Vieta's formula we have

$$x_1 + x_2 = p,$$
$$x_1 x_2 = -\frac{1}{2p^2}.$$

Hence we use the relationship between the arithmetic mean and geometric mean and we get

$$\begin{aligned} x_1^4 + x_2^4 &= (x_1 + x_2)^4 - 2x_1x_2(2(x_1+x_2)^2 - x_1x_2) \\ &= p^4 + \frac{1}{p^2}(2p^2 + \frac{1}{2p^2}) = 2 + p^4 + \frac{1}{2p^4} \\ &\geq 2 + \sqrt{p^4 \frac{1}{2p^4}} = 2 + \sqrt{2}. \end{aligned}$$

□

Solution The roots of the polynomial

$$x^2 - px - \frac{1}{2p^2}$$

are

$$x_1 = \frac{p + \sqrt{p^2 + \frac{2}{p^2}}}{2} \quad \text{and} \quad x_2 = \frac{p - \sqrt{p^2 + \frac{2}{p^2}}}{2}.$$

Hence

$$x_1^4 + x_2^4 = 2 + p^4 + \frac{1}{2p^4}.$$

Now we find the minimum of the function $f(x) = x^4 + \frac{1}{2x^4}$. The minimum occurs at the points $x = \pm 2^{-\frac{1}{8}}$. Hence $f(\pm 2^{-\frac{1}{8}}) = \sqrt{2}$ and we obtain

$$x_1^4 + x_2^4 \geq 2 + \sqrt{2}.$$

□

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Problem 3 *Prove that for all* $n \in \mathbb{N}$,

$$\prod_{i=1}^{n}\left(1+\frac{1}{2^i}\right) < 3\,.$$

Solution We prove that

$$\prod_{i=2}^{n}\left(1+\frac{1}{2^i}\right) < 2\,.$$

We know that $1+x < \frac{1}{1-x}$ for $0 < x < 1$. Hence

$$\left(1+\frac{1}{4}\right)\left(1+\frac{1}{8}\right)\cdots\left(1+\frac{1}{2^n}\right) < \frac{1}{(1-\frac{1}{4})(1-\frac{1}{8})\cdots(1-\frac{1}{2^n})}\,,$$

and because $(1-x)(1-y) > 1-x-y$ for $0 < x, y < 1$, we obtain

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\cdots\left(1-\frac{1}{2^n}\right) > 1-\frac{1}{4}-\frac{1}{8}-\ldots-\frac{1}{2^n} \geq \frac{1}{2}\,.$$

□

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Problem 4 *Decide whether there exists a non-constant function* $f\colon \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\big(f(x) - f(y)\big)^2 \le |x-y|^3 \tag{1}$$

for all $x, y \in \mathbb{R}$.

Solution From (1) we get

$$(\frac{f(x)-f(y)}{x-y})^2 \le |x-y| \, .$$

Thus

$$\lim_{y\to x} \frac{f(x)-f(y)}{x-y} = 0$$

and we have $f'(x) = 0$. Hence $f(x)$ is constant. □

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Category II

Problem 1 *Find a triple of integers* x, y, z, *each greater then* 50 *and satisfying*

$$x^2 + y^2 + z^2 = 3xyz\,. \tag{1}$$

Solution Let $x \leq y \leq z$. If (x, y, z) is a solution of the equation (1) then it is easy to check that $(y, z, 3yz - x)$ and $(x, z, 3xz - y)$ solve the equation too. The triple $(1, 1, 1)$ solve the same equation and hence is easy to find the triple (x, y, z) greater then 50 which solve the equation (1). $\square$

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Category II

Problem 2 *Prove that for an arbitrary $n \in \mathbb{N}$, the number*

$$\left(\frac{3+\sqrt{17}}{2}\right)^n + \left(\frac{3-\sqrt{17}}{2}\right)^n$$

is an odd integer.

Solution The numbers $\lambda_{1,2} = \frac{3\pm\sqrt{17}}{2}$ are the solutions of the equation $x^2-3x-2=0$, which is the characteristic equation of the recurrence $y_{n+2} = 3y_{n+1} + 2y_n$. We have $a_0 = 2$ and $a_1 = 3$. Then for $n \geq 1$

$$a_{n+2} = 3a_{n+1} + 2a_n \equiv 1a_{n+1} + 0a_n = a_{n+1} \pmod 2 .$$

Hence for all $n \geq 1$ the number

$$a_n = \left(\frac{3+\sqrt{17}}{2}\right)^n + \left(\frac{3-\sqrt{17}}{2}\right)^n$$

is an odd integer. □

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Problem 3 *Let the function* $f\colon \mathbb{R} \to \mathbb{R}$ *satisfy*

$$f(xy) = \frac{f(x) + f(y)}{x + y} \tag{1}$$

for all $x, y \in \mathbb{R}$, $x + y \neq 0$. *Is there* $x \in \mathbb{R}$ *such that* $f(x) \neq 0$?

Solution For $y = 1$ we have

$$f(x) = \frac{f(x) + f(1)}{x + 1} \quad (x \neq -1) \tag{2}$$

and for $y = 0$ we have

$$f(0) = \frac{f(x) + f(0)}{x} \quad (x \neq 0)\,. \tag{3}$$

From this equation we obtain $f(x) = f(0)(x - 1)$ and for $x = 1$ we get $f(1) = 0$. From (2) we obtain $xf(x) = 0$ and we have $f(x) = 0$ for all $x \neq 0, -1$. Now if we put $x = 2, y = 0$ into (1) we get $f(0) = 0$ and for $x = 0, y = -1$ we obtain $f(-1) = 0$. $\square$

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Problem 4 *How many real roots does the polynomial*

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^n}{n}$$

have?

Solution Let

$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^n}{n}\,.$$

We have two possibilities.

1. For n odd it is easy to check that $f'(x) > 0$ for all $x \in (-\infty, \infty)$. The function $f(x)$ is continuous and $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$, so we have one root.
2. For n even we obtain that $f'(x) < 0$ for $x \in (-\infty, -1)$, $f'(x) = 0$ for $x = -1$ and $f'(x) > 0$ for $x \in (-1, \infty)$. The function $f(x)$ has a minimum at the point $x = -1$, but $f(-1) > 0$ so $f(x)$ has no roots.

Hence the function $f(x)$ has at most one real root. □