The 4^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 6^{th} April 1994
Category I

Problem 1 *Prove that an arbitrary integer can be written as a sum of five cube powers of integers.*

Problem 2 *Prove that for the roots* x_1, x_2 *of the polynomial*

$$x^2 - px - \frac{1}{2p^2}\,,$$

where $p \in \mathbb{R}$ *and* $p \neq 0$, *the following inequality holds:*

$$x_1^4 + x_2^4 \geq 2 + \sqrt{2}\,.$$

Problem 3 *Prove that for all* $n \in \mathbb{N}$,

$$\prod_{i=1}^{n}\Big(1 + \frac{1}{2^i}\Big) < 3\,.$$

Problem 4 *Decide whether there exists a non-constant function* $f\colon \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\big(f(x) - f(y)\big)^2 \leq |x - y|^3 \qquad (1)$$

for all $x, y \in \mathbb{R}$.

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Category II

Problem 1 *Find a triple of integers* x, y, z, *each greater then* 50 *and satisfying*

$$x^2 + y^2 + z^2 = 3xyz\,. \tag{1}$$

Problem 2 *Prove that for an arbitrary* $n \in \mathbb{N}$, *the number*

$$\Big(\frac{3+\sqrt{17}}{2}\Big)^n + \Big(\frac{3-\sqrt{17}}{2}\Big)^n$$

is an odd integer.

Problem 3 *Let the function* $f\colon \mathbb{R} \to \mathbb{R}$ *satisfy*

$$f(xy) = \frac{f(x)+f(y)}{x+y} \tag{1}$$

for all $x, y \in \mathbb{R}$, $x + y \neq 0$. *Is there* $x \in \mathbb{R}$ *such that* $f(x) \neq 0$?

Problem 4 *How many real roots does the polynomial*

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^n}{n}$$

have?